

## ANOTHER LAW FOR 3-METABELIAN GROUPS

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**Abstract.** We show that  $[z, y]^{-1}[z, x]^{-1}[y, x]^{-1}[z, y][z, x][y, x] = 1$  is another defining law for the variety of 3-metabelian groups.

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A group  $G$  is defined to be metabelian if  $[G', G']$  is the trivial subgroup and is defined to be 3-metabelian if all of its three generator subgroups are metabelian. In 1956, Neumann [7] gave an example of a group that is 3-metabelian but is not metabelian. In 1961, Macdonald [4], among other results, obtained information about the structure of 3-metabelian groups and observed that such groups satisfy the law  $[x, y; x, z] = 1$ . In 1962, Macdonald [5] proved as a special case of Theorem 7 in his paper that any group that satisfies  $[x, y; x, z] = 1$  is 3-metabelian, and hence this law defines the variety of 3-metabelian groups. Of related interest, in 1964, Bachmuth and Lewin [1] proved that the law  $[x, y, z][y, z, x][z, x, y] = 1$  also defines the variety of 3-metabelian groups. Macdonald [6] was aware of this last result and proved, also in 1964, that the law  $[x, y; y, z][y, z; z, x][z, x; x, y] = 1$  is another law that defines the variety of 3-metabelian groups. We will use Macdonald's results to prove our result. The reader will find a discussion of these results and definitions for unexplained notation and terminology in Neumann's book [8].

The notation  $W(x, y, z)$  for  $[z, y]^{-1}[z, x]^{-1}[y, x]^{-1}[z, y][z, x][y, x]$  was introduced by Jackson, Gaglione and Spellman for expository convenience in [2] and used more extensively in [3]. In those papers, the following three properties of  $W(x, y, z)$  were used: for  $G$  any group and  $x, y, z$  any elements of  $G$ ,

$$\begin{aligned} [z, y, x] &= ([y, x, z]^{-1})^{[z, x][z, y]} W(x, y, z)[z, x, y]^{[y, x]}, \\ W(x, y, z) &= [z, y; z, x][z, y; y, x]^{[z, x]} [z, x; y, x] \text{ and} \\ W(x, y, z) &= [z, x; y, x]^{[z, y]} [z, y; y, x][z, y; z, x]^{[y, x]}. \end{aligned}$$

Jackson et al. were also aware of other identities, such as  $(W(y, x, z))^{[y, x]} = (W(x, y, z))^{-1}$ ,  $W(x, y, z) = (W(y, z, x))^{[z, x][y, x]}$  and  $W(x, x, z) = 1$ , but did not use or publish these.

**THEOREM.** *The variety of groups defined by the law  $W(x, y, z) = 1$  is the variety of 3-metabelian groups.*

*Proof.* Permuting variable names when necessary, and using the law  $[x, y; x, z] = 1$  from Macdonald's 1962 paper [5], the commutators  $[x, y]$ ,  $[x, z]$  and  $[y, z]$  commute with one another in any 3-metabelian group. Since  $W(x, y, z)$  is defined to be  $[z, y]^{-1}[z, x]^{-1}[y, x]^{-1}[z, y][z, x][y, x]$ , it is easy to see that  $W(x, y, z) = 1$  for any elements  $x, y$  and  $z$  of a 3-metabelian group.

To see that any group that satisfies the law  $W(x, y, z) = 1$  is 3-metabelian, we will use a result from Macdonald's 1964 paper [6]. It is proved there that the law  $[x, y; y, z][y, z; z, x][z, x; x, y] = 1$  defines the variety of 3-metabelian groups. We will show for any group  $G$  and arbitrary elements  $x, y$  and  $z$  in  $G$  that  $[x, y; y, z][y, z; z, x][z, x; x, y] = 1$  if  $W(x, y, z) = 1$ .

Using  $W(x, y, z) = 1$ , we see that

$$[z, y]^{-1}[z, x]^{-1}[y, x]^{-1} = [y, x]^{-1}[z, x]^{-1}[z, y]^{-1}.$$

Using this and the commutator identity  $[a, b] = [b, a]^{-1}$ , we obtain

$$[y, z][z, x]^{-1}[x, y] = [x, y][z, x]^{-1}[y, z]. \quad (1)$$

We next observe that  $[x, y; y, z][y, z; z, x][z, x; x, y]$  first expands by obvious substitutions to

$$([x, y]^{-1}[y, z]^{-1}[x, y][y, z]) ([y, z]^{-1}[z, x]^{-1}[y, z][z, x]) ([z, x]^{-1}[x, y]^{-1}[z, x][x, y]),$$

which reduces with obvious cancellations to

$$[x, y]^{-1}[y, z]^{-1}[x, y][z, x]^{-1}[y, z][x, y]^{-1}[z, x][x, y]. \quad (2)$$

We then use equation (1) to substitute  $[y, z][z, x]^{-1}[x, y]$  for the product of the third, fourth and fifth commutator factors in equation (2). We obtain

$$\begin{aligned} & [x, y; y, z][y, z; z, x][z, x; x, y] \\ &= [x, y]^{-1}[y, z]^{-1} ([x, y][z, x]^{-1}[y, z]) [x, y]^{-1}[z, x][x, y] \\ &= [x, y]^{-1}[y, z]^{-1} ([y, z][z, x]^{-1}[x, y]) [x, y]^{-1}[z, x][x, y], \end{aligned}$$

which then easily reduces to 1. □

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