COMMUTATOR CONDITIONS IN NORMAL CLOSURES OF ENGEL GROUPS

Christine Bussman, B.S.

An Abstract Presented to the Graduate Faculty of Saint Louis University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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Abstract

In this dissertation we explore connections between $n$-Engel groups and commutator properties of the normal closures of their elements. We explore examples where a group is $n$-Engel but an element of the group has a normal closure that is not nilpotent of class $n$. We also look at $n$-Engel groups where an element has a normal closure that is not $(n-1)$–Engel. Finally, we define and investigate a class of groups in which any $n$-Engel group contains only elements whose normal closures are $(n-1)$–Engel and nilpotent of class $n-1$. 
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A Dissertation Presented to the Graduate Faculty of Saint Louis University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy 2011
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    Chairperson and Advisor

Associate Professor Greg Marks

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Dedication

To Mark, Andrew, and Megan
Acknowledgements

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Chapter 1

Introduction and Preliminaries

1. Summary of Results

In this dissertation, we explore how the \( n \)-Engel condition for a group relates to the \( (n - 1) \)-Engel condition for the normal closure of every element of a group. We also see how these conditions relate to nilpotence of the normal closure of every element. The \( n \)-Engel condition and the normal closure of a group element are defined below. We first establish some standard notation.

**Definition 1.1.** If \( a \) and \( b \) are elements of a group \( G \), then we define the *conjugate* of \( a \) by \( b \), denoted by \( a^b \), to be \( b^{-1}ab \).

We will occasionally use the notation \( a^{-b} \) as an abbreviation for \( (a^{-1})^b \).

**Definition 1.2.** If \( a \) and \( b \) are elements of a group \( G \), then we define the *commutator* of \( a \) and \( b \), which we denote as \( [a,b] \), to be \( a^{-1}b^{-1}ab \).

In this dissertation, commutators are left normed, meaning the notation \( [a,b,c] \) refers to the commutator \( [[a,b],c] \). We also write \( [a,2b] \) for the commutator \( [a,b,b] \) and for \( n \) a positive integer greater than 2, we use the recursive notation \( [a,nb] \) for the commutator \( [a,_{n-1}b,b] \).

**Definition 1.3.** For any group \( G \), any \( X \) a subset of \( G \), and any positive integer \( n \), a *simple commutator* of weight \( n \) in \( X \) is a commutator of the form \( [x_1,\ldots,x_n] \), where each \( x_i \) is in \( X \).

If \( G \) is a group and \( n \) is an integer, we use the notation \( G^n \) for the subgroup \( \langle g^n \mid g \in G \rangle \). Also, if \( H \) and \( K \) are subgroups of \( G \), we use \( [H,K] \) to denote the subgroup \( \langle [h,k] \mid h \in H, k \in K \rangle \).
DEFINITION 1.4. Let $G$ be a group. The \textit{normal closure} in $G$ of an element $g$ of $G$, denoted by $g^G$, is defined as $\langle g^h \mid h \in G \rangle$.

DEFINITION 1.5. We define the \textit{lower central series} of a group $G$ to be the sequence $\gamma_1(G) \supseteq \gamma_2(G) \supseteq \ldots$ where $\gamma_1(G) = G$ and $\gamma_i(G) = [\gamma_{i-1}(G), G]$.

DEFINITION 1.6. A group $G$ is \textit{nilpotent} when for some positive integer $c$ the subgroup $\gamma_{c+1}(G)$ is trivial. In this case we call the smallest such $c$ the \textit{nilpotence class} of $G$.

DEFINITION 1.7. A group $G$ is \textit{locally nilpotent} when every finitely generated subgroup of $G$ is nilpotent.

DEFINITION 1.8. A group $G$ is called an \textit{Engel group} when for every $a, b \in G$, there is some positive integer $n$ such that $[a, n b] = 1$. If there is one positive integer $n$ such that $[a, n b] = 1$ for all pairs $a, b \in G$, then $G$ is an $n$-\textit{Engel group}.

We frequently need to calculate with commutators of group elements. The following lemma is necessary for these calculations. These identities are common knowledge, and are given without proof.

**Lemma 1.1.** Let $G$ be a group and let $a$, $b$, and $c$ be elements of $G$. Then

1. $[a b, c] = [a, c]^b[b, c] = [a, c][a, c, b][b, c]$.
2. $[a, b c] = [a, c][a, b]^c = [a, c][a, b][a, b, c]$.
3. $[a, b^{-1}] = [a, b]^{-b^{-1}} = [b, a]^{b^{-1}}$.
4. $[a^{-1}, b] = [a, b]^{-a^{-1}} = [b, a]^{a^{-1}}$.

We also will use the following lemma, which we will prove, although it is also common knowledge.

**Lemma 1.2.** Let $G$ be a group of nilpotence class $c$. If $x$ is an element of $\gamma_{c-1}(g)$ and $g$ and $h$ are elements of $G$, then $[x, g^h] = [x, g]$.
Proof. By Lemma 1.1,

\[
[x, g^h] = [x, gh][x, h^{-1}][x, h^{-1}, gh] \\
= [x, gh][x, h^{-1}] \\
= [x, h][x, g][x, g, h][x, h^{-1}] \\
= [x, h][x, g][x, h^{-1}] \\
= [x, g][x, h][x, h^{-1}] \\
= [x, g][x, h^{-1}h][x, h^{-1}, h]^{-1} \\
= [x, g].
\]

\[\square\]

Definition 1.9. If \( \mathcal{X} \) is a class of groups, then the class of groups \( \mathcal{L}(\mathcal{X}) \) is defined to be those groups \( G \) in which the normal closure of every element of \( G \) is in the class \( \mathcal{X} \). The property of being in the class \( \mathcal{L}(\mathcal{X}) \) is called the Levi property generated by \( \mathcal{X} \).

Let \( n \) be a positive integer. This dissertation explores how the following conditions relate:

\( \mathcal{E}_n \): A group \( G \) has the property \( \mathcal{E}_n \) if \( G \) is \( n \)-Engel.

\( \mathcal{L}(\mathcal{E}_n) \): A group \( G \) has the property \( \mathcal{L}(\mathcal{E}_n) \) if the normal closure of each element \( a \) of \( G \) is \( n \)-Engel.

\( \mathcal{L}(\mathcal{N}) \): A group \( G \) has the property \( \mathcal{L}(\mathcal{N}) \) if the normal closure \( a^G \) of each element \( a \) of \( G \) is nilpotent. A group that satisfies this property \( \mathcal{L}(\mathcal{N}) \) is often called a Fitting group.

\( \mathcal{L}(\mathcal{N}_n) \): A group \( G \) has the property \( \mathcal{L}(\mathcal{N}_n) \) if the normal closure \( a^G \) of each element \( a \) of \( G \) is nilpotent of class at most \( n \). Such a group is sometimes referred to as having Fitting degree \( n \).

\( \mathcal{B}_n \): A group \( G \) has the property \( \mathcal{B}_n \) if every subgroup that is maximal in the set of subgroups of \( G \) that are nilpotent of class at most \( n \) is also normal in \( G \). (Note that \( \mathcal{B}_n \) is used by some authors to denote a Burnside group.)
By abuse of notation, we will use the notations $E_n$, $L(E_n)$, $L(N_n)$, and $B_n$ to refer to both the conditions described and the classes of groups that satisfy the respective conditions.

We begin by proving simple implications among some of these conditions.

**Lemma 1.3.** Let $n$ be a positive integer. Then

1. If $G$ is a group that satisfies condition $B_n$, then $G$ also satisfies condition $L(N_n)$.
2. If $G$ is a group that satisfies condition $L(N_n)$, then $G$ also satisfies condition $L(E_n)$.
3. If $G$ is a group that satisfies condition $L(E_n)$, then $G$ also satisfies condition $E_{n+1}$.

**Proof.**

(1) Suppose that $G$ is a group that satisfies $B_n$. Let $g$ be an element of the group $G$. Let $\mathcal{S}$ be the set of subgroups of $G$ that contain $g$ and are nilpotent of class at most $n$. The class of groups $\mathcal{S}$ is not empty because it contains $\langle g \rangle$. Let $N$ be a maximal element of $G$. By hypothesis, $N$ is normal in $G$. Thus $g^G \subset N$. Hence $g^G$ is nilpotent of class at most $n$, so $G$ also satisfies $L(N_n)$.

(2) Suppose that $G$ is a group that satisfies $L(N_n)$. Let $g$ be an element of $G$ and let $x$ and $y$ be elements of $g^G$. The $n$-Engel commutator $[x, ny]$ is a commutator of length $n + 1$ in $g^G$. Because $g^G$ is nilpotent of class $n$, we see that $[x, ny]$ is trivial and so $g^G$ is $n$-Engel. Thus, $G$ also satisfies condition $L(E_n)$.

(3) Let $G$ be a group that satisfies condition $L(E_n)$. If $a$ and $b$ are elements of $G$, then $[a, n+1b] = [[a, b], n b]$. Because $[a, b]$ is in $b^G$, the commutator $[a, n+1b]$ is trivial, so $G$ is $(n + 1)$-Engel.

$\square$
Lemma 1.3 shows that for any positive integer $n$, condition $\mathcal{B}_n$ implies condition $\mathcal{L}(\mathcal{N}_n)$, which implies condition $\mathcal{L}(\mathcal{E}_n)$, which in turn implies condition $\mathcal{E}_{n+1}$. As we see in more detail in our discussion of the history of Engel groups in Section 3 of Chapter 1, it is known that conditions $\mathcal{B}_1$, $\mathcal{L}(\mathcal{N}_1)$, $\mathcal{L}(\mathcal{E}_1)$, and $\mathcal{E}_2$ are equivalent because $\mathcal{E}_2$ implies $\mathcal{B}_1$ [1]. Similarly, $\mathcal{B}_2$ is stronger than $\mathcal{L}(\mathcal{N}_2)$, $\mathcal{L}(\mathcal{E}_2)$, and $\mathcal{E}_3$, but $\mathcal{L}(\mathcal{N}_2)$, $\mathcal{L}(\mathcal{E}_2)$, and $\mathcal{E}_3$ are equivalent because condition $\mathcal{E}_3$ implies $\mathcal{L}(\mathcal{N}_2)$ [2]. It is known that condition $\mathcal{L}(\mathcal{N}_3)$ is stronger than condition $\mathcal{E}_4$ [3]. It is also known that condition $\mathcal{E}_4$ is equivalent to condition $\mathcal{L}(\mathcal{E}_3)$, but that condition $\mathcal{E}_5$ is not equivalent to condition $\mathcal{L}(\mathcal{E}_4)$ [4]. Also, $\mathcal{E}_5$ does not imply $\mathcal{L}(\mathcal{N})$ [3]. We will look at these situations in more detail in Sections 3 and 4 of Chapter 1.

We explore the connections between condition $\mathcal{L}(\mathcal{N}_n)$ and condition $\mathcal{E}_{n+1}$ in Chapter 3. In Chapter 3, Section 1 we explore the details of an example by Gupta and Levin [3]. Their example shows the following theorem.

**Theorem 3.5.** (Gupta and Levin [3]) There exists a 4-Engel 5-group $H$ with an element $a$ whose normal closure $a^H$ is not nilpotent of class 3.

We also show how this example can be investigated using GAP, and obtain the fact that this example is of order at most $5^{12}$.

In Section 2 of Chapter 3, we find some limitations on groups that satisfy condition $\mathcal{E}_4$ but not condition $\mathcal{L}(\mathcal{N}_3)$. The following theorems summarize our results.

**Theorem 3.8.** If $G$ is a 4-Engel group and is nilpotent of class 5, then the normal closure in $G$ of every element of $G$ is nilpotent of class at most 3.

**Theorem 3.9.** If $G$ is a 4-Engel group on two generators with exponent 5, then for every element $a$ in $G$, the normal closure $a^G$ is nilpotent of class at most 3.

Chapter 3, Section 3 discusses an example, published by Nickel [5], of a 4-Engel 2-group that has a normal closure that is not nilpotent of class 3. Section 4 of
Chapter 3 gives a new example of a 4-Engel 2-group that has a normal closure which is not nilpotent of class 3 and that is smaller than Nickel’s example.

In Section 5 of Chapter 3, we give new examples that show that condition $E_{n+1}$ is not equivalent to condition $L(N_n)$ for other values of $n$ besides 4, and also that 2 and 5 are not the only primes for which $p$-groups satisfying condition $E_{n+1}$ need not satisfy condition $L(N_n)$. We summarize our results in the following two theorems.

**Theorem 3.11.** There exist 5-Engel and 6-Engel 5-groups that do not satisfy conditions $L(N_4)$ and $L(N_5)$ respectively. There also exist 5-Engel and 6-Engel 2-groups that do not satisfy conditions $L(N_4)$ and $L(N_5)$ respectively.

**Theorem 3.12.** There exist 5-Engel 3-groups and 7-groups that do not satisfy condition $L(N_4)$.

In Chapter 4 we examine another example by Gupta and Levin from [3]. This example shows that condition $E_n$ does not always imply condition $L(N)$. It proves the following theorem:

**Theorem 4.2.** (Gupta and Levin [3]) For every odd prime $p$, there exists a $p$-group that is $(p + 2)$-Engel, but that has an element whose normal closure is not nilpotent.

In Chapter 5 we consider how conditions $L(E_n)$ and $E_{n+1}$ are related. In Section 1 of 5, we consider an example by Vaughan-Lee [4] of a 3-group that satisfies condition $E_5$ but not $L(E_4)$. We also discuss a smaller 3-group with the same properties.

In Section 2 of Chapter 5, we examine the details of an example published by Rips and Shalev [6], which demonstrates the following theorem.

**Theorem 5.15.** (Rips and Shalev [6]) For every prime $p \neq 2$, there is some positive integer $n$ for which there is an $n$-Engel $p$-group that has an element whose normal closure in the whole group is not $(n - 1)$-Engel.
In their paper, Rips and Shalev comment that a similar construction is possible for the prime 2. In Chapter 5, Section 2.4 we work through the construction of such a 2-group, resulting in the following theorem.

**Theorem 5.21.** For some positive integer $n$, there is an $n$-Engel 2-group that has an element whose normal closure in the whole group is not $(n - 1)$-Engel.

In Chapter 5, Section 3 we calculate upper bounds for the smallest values of $n$ in Theorem 5.15 and Theorem 5.21.

Morse [7] proves the following:

**Theorem 1.4.** (Morse)

1. (Theorem 2 in [7]) Let $n \geq 3$ be a natural number and let $G$ be a group of nilpotency class $n + 2$. Then $G$ is $(n + 1)$-Engel if and only if the normal closure in $G$ of every element of $G$ is nilpotent of class $n$ or less.

2. (Theorem 3 in [7]) Let $n \geq 1$ be a natural number and let $G$ be a group that satisfies the law $[x_1, x_2, x_3, [x_4, x_5], x_6] = 1$ for all $x_i \in G$, $i = 1, \ldots, 6$. Then $G$ is $(n + 1)$-Engel if and only if the normal closure in $G$ of every element of $G$ is nilpotent of class $n$ or less.

3. (Theorem 4 in [7]) Let $n \geq 4$ be a natural number and let $G$ be a group that satisfies the laws $[x_1, x_2, x_3, [x_4, x_5], x_6, x_7] = 1$, $[x_1, x_2, x_3, [x_4, x_5], x_6, x_7] = 1$, and $[x_1, x_2, [x_3, x_4], [x_5, x_6, x_7] = 1$ for all $x_i \in G$, $i = 1, \ldots, 7$. Then $G$ is $(n + 1)$-Engel if and only if the normal closure in $G$ of every element of $G$ is nilpotent of class $n$ or less.

In Chapter 6 we generalize Theorem 1.4 to find a class of groups, which we will call $\mathfrak{W}_m$, in which conditions $\mathfrak{E}(\mathfrak{M}_n)$ and $\mathfrak{E}_{n+1}$ are equivalent. For a positive integer $n$, we consider a group to be a $\mathfrak{W}_n$ group if for every $i$ in $\{3, \ldots, n - 1\}$ the group satisfies the law $[x_1, \ldots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \ldots, x_n] = 1$ for all $x_i \in G$.

**Theorem 6.7.** Let $m$ be an integer greater than or equal to 4. If $G$ is a $\mathfrak{W}_m$ group, then for $n \geq m - 2$, the following are equivalent:
(1) $x^G$ is nilpotent of class at most $n$ for every $x \in G$.

(2) $x^G$ is $n$-Engel for every $x \in G$.

(3) $G$ is $(n+1)$-Engel.

2. Definitions

In this section, we collect some definitions that we will use.

**Definition 1.10.** If $G$ and $H$ are two groups and $\alpha : H \to \text{Aut}(G)$ is a homomorphism from $H$ into $\text{Aut}(G)$, we define the *semidirect product* $G \rtimes H$ to be the set $\{(g, h) \mid g \in G, h \in H\}$ with the binary operation

$$(g_1, h_1)(g_2, h_2) = (g_1\alpha(h_1)(g_2), h_1h_2).$$

In the semidirect product $G \rtimes H$, the subgroup $\{(g, 1) \mid g \in G\}$ is normal and isomorphic to $G$. Sometimes, by abuse of notation, we will consider $\{(g, 1) \mid g \in G\}$ to be $G$. We will usually write the homomorphism action as conjugation, in which case the product is written as $(g_1, h_1)(g_2, h_2) = (g_1g_2^{h_1}, h_1h_2)$. We will also use the fact that $G$ is normal in $G \rtimes H$. The homomorphism $\alpha$ is sometimes explicitly referenced by the notation $G \rtimes_\alpha H$, but we sometimes do not specify the automorphism in the notation. The semidirect product of two groups is again a group.

We next define the wreath product of two groups $G$ and $H$.

**Definition 1.11.** Let $G$ and $H$ be groups. We define the base group $B$ of the wreath product by

$$B = \prod_{h \in H} G_h,$$  \hspace{1cm} (2)

where each $G_h$ is an isomorphic copy of $G$. We use the notation $g_h$, where $g \in G$ and $h \in H$, to represent the copy of $g$ that is present in $G_h$. Consider the homomorphism $\alpha : H \to \text{Aut}(B)$ defined componentwise by $\alpha(h^*)(g_h) = g_{hh^*}$ for each $h^*$ in $H$.

Define the *wreath product* $G \wr H$ to be the semidirect product $B \rtimes_\alpha H$. 

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Definition 1.12. A subgroup $H$ of a group $G$ is called subnormal in $G$ if there exists a series of subgroups $H = J_0 \lhd J_1 \lhd \cdots \lhd J_n = G$. Such a series of subgroups is called a subnormal series for $H$, and is said to have length $n$. We say $H$ is subnormal of defect $m$ when $H$ is subnormal in $G$ and $m$ is the length of the shortest subnormal series for $H$.

Definition 1.13. The derived series of group $G$ is a recursively defined series of subgroups of $G$ that are defined as follows. Let $G^{(0)} = G$ and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for $i \in \{1, 2, \ldots \}$. A group $G$ is solvable if there is some positive integer $n$ such that $G^{(n)} = 1$. If $n$ is the least such integer, then $G$ is solvable of derived length $n$.

Definition 1.14. A group $G$ is a periodic group, also called a torsion group, if every element of $G$ has finite order.

Definition 1.15. A group $G$ is torsion free if $G$ does not contain any nontrivial elements of finite order.

Definition 1.16. A group $G$ is residually finite if for every non-trivial element $g$ of $G$ there is a normal subgroup $H_g$ of finite index in $G$ that does not contain $g$.

Definition 1.17. If $p$ is an odd prime and $G$ is a $p$-group, then $G$ is called a powerful $p$-group if $[G, G]$ is contained in $G^p$. If $G$ is a 2-group, then $G$ is a powerful 2-group if $[G, G]$ is contained in $G^4$.

Definition 1.18. A group $G$ is compact if it is considered as a topological group, and its topology is compact.

Definition 1.19. A group $G$ is right orderable if there exists a total order relation $\geq$ on $G$ such that if $a, b, g$ are elements of $G$, then $a \geq b$ implies $ga \geq gb$.

Definition 1.20. An associative algebra over a field $F$ is a set $A$ with two binary operations $+$ and $\cdot$, referred to as addition and multiplication, and an action of $F$ on $A$ such that
1. A with respect to + is an abelian group;
2. A is a semigroup with respect to multiplication;
3. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ for every $a, b, c \in A$;
4. A is a vector space over $F$; and
5. the action of $F$ on $A$ is compatible with multiplication in $A$ in the sense that $f(a \cdot b) = (fa) \cdot b = a \cdot (fb)$ for $f \in F$ and $a, b \in A$.

**Definition 1.21.** A **Lie ring** $R$ is an abelian group with an operator $(-, -)$ such that

1. $(a + b, c) = (a, c) + (b, c)$ and $(a, b + c) = (a, b) + (b, c)$ for every $a, b, c \in R$;
2. $(b, b) = 0$ for every $b \in R$; and
3. $((a, b), c) + ((b, c), a) + ((c, a), b) = 0$ for every $a, b, c \in R$.

**Definition 1.22.** Let $R$ be a Lie ring. We define the **derived series** of $R$ to be the sequence of subrings $R^{(i)}$ for $i$ in $\{0, 1, 2, \ldots \}$, where $R^{(0)} = R$ and $R^{(1)} = (R, R)$ and $R^{(i)} = (R^{(i-1)}, R^{(i-1)})$ for $i > 1$. We say the Lie ring $R$ is **Lie-solvable** if $R^{(i)} = 0$ for some positive integer $i$.

**Definition 1.23.** A **Lie algebra** $L$ over a field $F$ is a Lie ring over $F$ that satisfies the additional properties $(fa + gb, c) = f(a, c) + g(b, c)$ and $(a, fb + gc) = f(a, b) + g(a, c)$ for every $a, b, c \in L$ and for every $f, g \in F$.

**Definition 1.24.** For a Lie algebra $L$, the operator $(-, -)$ is referred to as a **Lie bracket**.

In this dissertation, the Lie bracket is left normed, so $(a, b, c)$ is used to represent $((a, b), c)$. Further, we recursively define the Lie bracket $(a, n b)$ as $(a, n_{-1} b, b)$, where $(a, 1 b)$ is defined as $(a, b)$. Also, if $M$ and $N$ are subalgebras of a Lie algebra $L$, we use the notation $(M, N)$ to represent the subalgebra generated by the set $\{(m, n) | m \in M, n \in N \}$.

The Lie algebras we consider in this dissertation are almost all related to an associative algebra.
**Definition 1.25.** Any associative algebra $A$ has a related Lie algebra $L$. The Lie algebra has the same elements as $A$, with the Lie bracket given by $(a, b) = ab - ba$. In this case, the Lie bracket is called the **Lie commutator**.

**Definition 1.26.** Let $L$ be a Lie algebra. We define the derived series of $L$ to be the sequence of subalgebras $L^{(i)}$ for $i$ in $\{0, 1, 2, \ldots \}$, where $L^{(0)} = L$ and $L^{(1)} = (L, L)$ and $L^{(i)} = (L^{(i-1)}, L^{(i-1)})$ for $i > 1$. We say the Lie algebra $L$ is **solvable** if $L^{(i)} = 0$ for some positive integer $i$.

**Definition 1.27.** Let $L$ be a Lie algebra. We define the lower central series of $L$ to be the sequence of subalgebras $L^i$ for $i$ in $\{1, 2, \ldots \}$ given by $L^1 = L$, $L^2 = (L, L)$ and $L^i = (L^{i-1}, L)$ for $i > 2$. We say the Lie algebra $L$ is **nilpotent** if $L^i = 0$ for some positive integer $i$.

**Definition 1.28.** Let $L$ be a Lie algebra. If for every $a, b \in L$ there is a positive integer $n$ such that $(a, n b)$ is trivial, then $L$ is an **Engel Lie algebra**. If there is some positive integer $n$ such that $(a, n b) = 0$ for every $a, b \in L$, then $L$ is an $n$-**Engel Lie algebra**.

**3. History of Engel Groups**

In this section we examine the history of Engel groups in general. In the next section, we consider the history of $n$-Engel groups in particular.

The groups now called Engel groups have become an area of interest in their own right, but their history is not well known beyond the fact that they are “related” to Lie algebras.

The earliest known reference to groups that are later called Engel groups occurs in the work of Burnside in 1902 [8]. Burnside’s paper refers to “groups where any two conjugate operations commute,” which Lemma 1.5 shows is equivalent to being a 2-Engel group.
Lemma 1.5. A group $G$ is a 2-Engel group if and only if any two conjugate elements of $G$ commute.

Proof. Any two conjugate elements commute in group $G$ exactly when for any $a, b \in G$, we know $[a^b, a] = 1$. However,

$$[a^b, a] = [a[a, b], a]$$
$$= [a, b, a]$$
$$= [[b, a]^{-1}, a]$$
$$= [b, a, a]^{-[b, a]^{-1}}.$$

Thus $[a^b, a]$ is trivial exactly when $[b, a, a]$ is trivial. \qed

Also in 1902, Fite [9] defined “class” in a way that is equivalent to the usual definition of nilpotence class, although nilpotence in groups was not defined until the 1930s.

In 1909, Fite [10] showed that 2-Engel groups (using terminology similar to that of Burnside) are nilpotent of class at most 3. By providing an example of a 2-Engel group of class exactly 3, Fite showed that 3 is the best possible bound on the nilpotence class of a 2-Engel group.

It seems that current interest in Engel groups comes from questions about Lie algebras. The first result of note in this direction is an unpublished result of van Kampen that finite dimensional Engel Lie algebras over fields of characteristic 0 are nilpotent. This result is referenced in Jacobson [11], who says that it was “found by Dr. van Kampen, Hamburg, 1928, but has not been previously published.”

In 1929 Hopkins [12] also showed that finite 2-Engel groups are nilpotent of class at most 3. He refers to the work of Burnside [8], but not the work of Fite [10]. His terminology is also of interest because he refers to 2-Engel groups as groups where “conjugate elements commute,” which is closer to our current usage than that of Burnside.
The same result as that of van Kampen was also proved by Weyl in class lectures (also cited by [11]). Jacobson [11] proved that finitely generated Engel Lie algebras over fields of characteristic 0 are nilpotent.

Bouton [13] states that Magnus and Zassenhaus drew the connection between Lie algebras and groups in the 1930s. Papers Magnus published during this time frame that include this connection are [14], [15], and [16]. Zassenhaus also used this connection in his 1940 paper [17], in which he shows that finite Engel groups are nilpotent and in which he connects Lie algebras with finite groups. On page 4 of [17], Zassenhaus takes credit for the label “nilpotent” for groups. Zassenhaus also lists and proves some analogous conditions for groups and Lie algebras.

In the abstracts [18] for an AMS conference on June 18, 1936, M. Zorn describes two talks that are of interest. In one of these ([19]), he proves that any finite Engel group is nilpotent, and in the other [20], that a finite Engel Lie algebra over a field of characteristic 0 is nilpotent (he refers erroneously to this result as related to a theorem of Lie, rather than of Engel, as he points out later [21]). In Zorn’s work the connections between Lie algebras and groups are also apparent, because the concepts of “Engel” and “nilpotent” are used for groups, although they were first used for rings and algebras. Zorn explicitly says in the abstract about groups ([19]) that the group abstract is the analogue of the abstract about algebras ([20]). The next year, he published another paper [21] in which he shows that \( n \)-Engel Lie rings with the maximal condition on subrings are nilpotent. He comments that this paper replaces one of the abstracts from the previous year.

In 1940, Baer [22] provided examples of non-nilpotent \( n \)-Engel groups. Baer does not explicitly state that his examples have these properties, but he does state that he provides for each prime \( p \) an infinite \( p \)-group that has trivial center but abelian commutator subgroup. These groups (Example 3.4 in [22]) are obtained by manipulating \( p \)-adic expansions of integers, but as Gupta points out ([23], page 32)
they are equivalent to wreath products $C_p \wr (\bigoplus_{i=1}^{\infty} C_p)$. Cohn [24] notes that Baer’s examples are $n$-Engel but not nilpotent.

In 1942, Levi [25] proved the following:

**Theorem 1.6.** Let $G$ be a 2-Engel group. Then

1. $G$ is nilpotent of class $\leq 3$.
2. If $a$, $b$, and $c$ are elements of $G$, then $[a, b, c]^3 = 1$.
3. $[a, b, c] = [b, c, a]$ for all $a, b, c$ elements of $G$.

The first part of Theorem 1.6 was already shown for finite groups in [12], but it is Levi’s result that is usually cited now. Meier-Wunderli [26] proved these same results in 1949 and pointed to Fite’s example of a 2-Engel group of class exactly 3 to show that 3 is the best possible bound on the nilpotence class of a 2-Engel group.

In 1953 Gruenberg [27] showed that a finitely generated solvable Engel group is nilpotent, and that a finitely generated solvable Engel Lie ring is nilpotent. Gruenberg’s paper is of particular note because it is the earliest paper the author of this dissertation can locate in which the term “Engel group” is used. M. Zorn and others before this discuss Engel groups, but do not use the term “Engel group.” Cohn [24] gives Gruenberg credit for first using the term “Engel group,” but does not specify where he does so.

In 1954 Cohn [24] provided examples of a non-nilpotent Engel group and a non-nilpotent Engel ring. He credits Gruenberg with an unpublished example of such a group (his exact wording is “Dr. Gruenberg has informed me that he also has such an example”), and adds a note that after writing his paper he was made aware of Baer’s 1940 example. Cohn starts with $F$ a field of characteristic $p$, where $p$ is a prime, and then constructs $K$, a purely transcendental extension of $F$ of countable (infinite) transcendence degree. He then constructs $M$, a free left $K$-module on an infinite basis, and considers $K$ to be the free $K$-module on the generator 1, where 1 is the unit element of $K$. The associative algebra $A$ is then
constructed as a sum $K + M$, and Cohn imposes a Lie multiplication on $A$, getting a Lie algebra $L$. He shows that $L$ is $(p + 1)$-Engel, but not nilpotent. Cohn’s group example $G$ is generated by elements of the form $\alpha(1 + x)$, where $\alpha$ is a nonzero element of $K$ and $x$ is an element of $M$. This group is isomorphic to $C_p \wr (\oplus_{i=1}^\infty C_p)$. The group $G$ is $(p + 1)$-Engel but not nilpotent.

In 1957, Baer [28] defined a group to be Noetherian if every subgroup is finitely generated, and proved that Engel groups that are Noetherian are also nilpotent.

Golod [29] (translated [30]) constructed a $p$-group that is finitely generated, residually finite, and not locally nilpotent. He does so by considering the multiplicative group generated by $\{1 + x_i\}$, where the $x_i$ are generators of a particular polynomial ring over the field $Z_p$. He comments that the same method of construction can be used to give a non-nilpotent finitely generated Engel group, thus proving that not all Engel groups are locally nilpotent.

To add to the information about Engel $\mathfrak{X}$-groups for various group-theoretic properties $\mathfrak{X}$, in 1960 Garaščuk and Supranenko [31] showed that a linear group over a field of characteristic zero is Engel if and only if it is locally nilpotent.

In 1998, Burns and Medvedev [32] showed that the commutator subgroup of a finitely generated Engel group is also finitely generated.

In 2003 Medvedev [33] proved that a compact Engel group is locally nilpotent.

4. Recent History of $n$-Engel Groups

In this section we examine more recent historical developments in the study of $n$-Engel groups, organized by property. The organization of this section is topical rather than chronological.

4.1. General $n$-Engel Groups. In 1991 Wilson [34] showed that a finitely generated residually finite $n$-Engel group is nilpotent.

In 2002, H. Smith [35] proved that every $n$-Engel group in which every subgroup is subnormal is nilpotent. In the same year, Abdollahi and Traustason
[36] showed that for every positive integer \( n \), there is a positive integer \( s(n) \) such that every powerful \( n \)-Engel \( p \)-group is nilpotent of class at most \( s(n) \).

Traustason and Crosby will soon publish a paper [37] in which they show that there exist positive integers \( m(n) \) and \( r(n) \) such that the law \([x^{r(n)}, x_1, \ldots, x_{m(n)}] = 1\) holds for all \( x \) and for all \( x_i, i = 1, \ldots, m(n) \) in any locally nilpotent \( n \)-Engel group.

In 1961, Gruenberg [38] showed that every \( n \)-Engel group that is solvable of derived length \( d \) and that has no elements of prime order less than \( n \) is nilpotent of class at most \((1 + 2^{n-2})d/2^{n-2}\).

4.2. One-Engel Groups. A group that is 1-Engel is abelian. Abelian groups have their own theory that is not in the scope of this dissertation.

4.3. Two-Engel Groups. A 2-Engel group is, in some sense, almost abelian. In 1961, W.P. Kappe [1] showed that condition \( \mathcal{E}_2 \) implies condition \( \mathcal{B}_1 \), which means that conditions \( \mathcal{B}_1 \), \( \mathcal{L}(\mathfrak{N}_1) \), \( \mathcal{L}(\mathcal{E}_1) \), and \( \mathcal{E}_2 \) are equivalent. In particular, if \( a \) is an element of a 2-Engel group \( G \), then \( a^G \), the normal closure of \( a \) in \( G \), is abelian. Another well-known result (see Theorem 1.6) is that any 2-Engel group is nilpotent of class at most 3, and is nilpotent of class at most 2 if there are no elements of exponent 3 in the group by part (2) of Theorem 1.6. Other important papers that discuss 2-Engel groups are by Burnside [8], Fite [9] [10], and Levi [25], and are discussed in Section 3 above.

4.4. Three-Engel Groups. Heineken [39] proved that all 3-Engel groups are locally nilpotent, and that if a 3-Engel group \( G \) has no elements of order 2 or 5, then \( G \) is nilpotent of class at most 4. If \( G \) has an element of order 2 but not of order 5, then \( G \) is solvable (N. Gupta, [40]) but not necessarily nilpotent (Baer, [22]). Bachmuth and Mochizuki [41] in 1971 gave an example of a 3-Engel group of exponent 5 that is not solvable, let alone nilpotent.

Heineken [42] in 1971 described a subclass of 3-Engel groups that have properties like those described by Levi and others for 2-Engel groups. Heineken’s
paper considers those 3-Engel groups whose cyclic subgroups are subnormal of defect 2, and shows that these groups are nilpotent of class at most 5, that weight 4 commutators in these groups have exponent 3, and that \([a, [b, [b, c]]]] = 1\) for all elements \(a, b,\) and \(c\) of such groups. All 2-Engel groups satisfy all three of these conditions.

In [2] W.P. Kappe and L.-C. Kappe showed that \(E_3\) implies \(L(N_2)\), so conditions \(L(N_2), L(E_2),\) and \(E_3\) are equivalent. Hence \(G\) is a 3-Engel group if and only if for every element \(a\) of \(G\), the nilpotence class of \(a^G\) is 2 or smaller. In 1989 N. Gupta and Newman [43] showed that a 3-Engel \(n\)-generator group \(G\) is nilpotent of class at most \(2n - 1\) if \(n > 2\), that a 3-Engel group with no elements of order 5 has nilpotence class at most \(n + 2\), and that the exponent of the 5th term of the lower central series of a 3-Engel group divides 20. Heineken in 1961 [39] showed that a 3-Engel group that has no elements of order 2 or 5 is nilpotent of class at most 4.

4.5. Four-Engel Groups. There are many results about 4-Engel groups, and for this reason the results in this subsection are given ordered by importance for the content of this dissertation rather than chronologically.

In 2005, Traustason [44] gave some properties of 2-generator 4-Engel groups. Later in 2005, Havas and Vaughan-Lee [45] proved that all 4-Engel groups are locally nilpotent, although their proof was highly dependent on computer computations in a specific group. In that same year, Traustason [46] published results that replace Havas and Vaughan-Lee’s computer calculations with hand calculations, effectively making the proof of Havas and Vaughan-Lee computer free.

In 1980, Gupta and Levin [3] constructed a 4-Engel 5-group that has an element whose normal closure is not nilpotent of class 3, thus showing that \(E_4\) does not imply \(L(N_3)\). We examine this example in detail in Chapter 3. In 1999, Nickel [5] constructed a 2-group example similar to the example of Gupta and Levin for 5-groups. In 2003, Traustason [47] proved that the normal closure of every element
of a 4-Engel group without elements of order 2 or 5 is nilpotent of class 3. In 2007, Vaughan-Lee [4] proved that condition $E_4$ implies condition $L(E_3)$.

Traustason [48] showed in 1994 that if a 4-Engel group has no elements of order 2, 3, or 5, then it is nilpotent of class at most 7. In 2002, Abdollahi and Traustason [36] showed that a 4-Engel group without elements of order 2 or 5 is solvable, but a 4-Engel group with elements of order 5 is not always solvable because Bachmuth and Mochizuki [41] construct a 3-Engel group with elements of order 5 that is not solvable. Abdollahi and Traustason [36] point out that a 4-Engel 2-group is not necessarily solvable because every group of exponent 4 is central-by-4-Engel (Bayes, Kautsky, and Wamsley [49]) and Razmyslov [50] (translated into English as [51]) constructs a non-solvable group of exponent 4 (note that if a group is not solvable, then the quotient of that group by its center remains non-solvable).

Longobardi and Maj [52] proved in 1997 that every right orderable 4-Engel group is nilpotent.

4.6. $n$-Engel Groups for Larger Values of $n$. It is not known in the literature if $E_5$ implies $L(N_4)$. This question will be resolved in Section 5 of Chapter 3, where we show that there are 2-groups, 3-groups, 5-groups, and 7-groups that satisfy Condition $E_5$ but not Condition $L(N_4)$. In [4], Vaughan-Lee constructed a 5-Engel group that has an element whose normal closure is not 4-Engel, proving that condition $E_5$ does not imply condition $L(E_4)$. Vaughan-Lee’s example is a 3-group.

It is not known if for other primes $p$ there exist $p$-groups that satisfy condition $E_5$ but not condition $L(E_4)$. It is also not known if 5-Engel groups are locally nilpotent.

For values of $n$ greater than 5, groups that are $n$-Engel have not been extensively studied. In Section 5 of Chapter 3, we show that there are groups that are 6-Engel but contain an element whose normal closure is not nilpotent of class 5. Rips and Shalev [6] show that for every odd prime $p$, there is some value of $n$ such that for $p$-groups, $E_n$ does not imply $L(E_{n-1})$. The result of Rips and Shalev will be discussed in detail in Chapter 5. Gupta and Levin [3] also show that for every odd
prime $p$, there is a $p$-group that is $(p + 2)$-Engel but that does not satisfy condition $\mathcal{L}(\mathfrak{N})$. These examples of Gupta and Levin are explained in detail in Chapter 4. It is not known if $n$-Engel groups are locally nilpotent for values of $n$ larger than 5.

5. Further Questions

In Chapter 4, we show that not every $n$-Engel group has property $\mathcal{L}(\mathfrak{N})$. It would be interesting to know:

**Question 1.** Does every finitely generated $n$-Engel group satisfy condition $\mathcal{L}(\mathfrak{N})$?

**Question 2.** Does every torsion-free $n$-Engel group satisfy condition $\mathcal{L}(\mathfrak{N})$?

In Chapter 5, for each prime $p$ we find an upper bound on the least $n$ such that a $p$-group with property $\mathfrak{E}_n$ need not have property $\mathcal{L}(\mathfrak{E}_{n-1})$. It would be interesting to resolve:

**Problem 3.** For each prime $p$, find an improved bound for the least $n$ such that a $p$-group with property $\mathfrak{E}_n$ need not have property $\mathcal{L}(\mathfrak{E}_{n-1})$.

The groups in Chapter 5 are not solvable, and are torsion groups. It would be of interest to answer the following questions:

**Question 4.** Is condition $\mathfrak{E}_n$ equivalent to condition $\mathcal{L}(\mathfrak{E}_{n-1})$ for torsion-free groups?

**Question 5.** Is $\mathfrak{E}_n$ equivalent to condition $\mathcal{L}(\mathfrak{E}_{n-1})$ for solvable groups?

In Chapter 6, we show that in the variety $\mathfrak{W}_m$, Conditions $\mathfrak{E}_n$, $\mathcal{L}(\mathfrak{E}_{n-1})$, and $\mathcal{L}(\mathfrak{N}_{n-1})$ are equivalent when $n \geq m - 2$. This suggests the following questions:

**Question 6.** Is Condition $\mathfrak{B}_n$ equivalent to Conditions $\mathfrak{E}_n$, $\mathcal{L}(\mathfrak{E}_{n-1})$, and $\mathcal{L}(\mathfrak{N}_{n-1})$ for $\mathfrak{W}_m$ when $n \geq m - 2$?
QUESTION 7. If the answer to Question 6 is no, is there another variety in which Condition $\mathfrak{B}_n$ is equivalent to Conditions $\mathfrak{E}_n$, $\mathfrak{L}(\mathfrak{E}_{n-1})$, and $\mathfrak{L}(\mathfrak{N}_{n-1})$ for $\mathfrak{M}_m$ when $n \geq m - 2$?

Finally, there still remains open the question:

QUESTION 8. Are $n$-Engel groups locally nilpotent for values of $n$ larger than 4?
Chapter 2

Computer Background and Methodology

In this chapter we introduce some background that is needed for computer calculations in this dissertation.

1. Power Commutator Presentations

In this section, we define a presentation for a group of prime power order called a power-commutator presentation. This presentation is useful for computer computations, because, as we will see, it provides a normal form for elements of the group.

Definition 2.1. Let \( G \) be a finite \( p \)-group, where \( p \) is a prime number. A power-commutator presentation (usually shortened to pc presentation) of \( G \) is a presentation of the form
\[
\langle a_1, \ldots, a_n | a_i^p = v_{ii}, 1 \leq i \leq n, [a_k, a_j] = v_{jk}, 1 \leq j < k \leq n \rangle,
\]
where \( v_{jk} \) is a word in the elements \( a_{k+1}, \ldots, a_n \). The generators \( a_1, \ldots, a_n \) of a pc presentation for \( G \) are referred to as pc generators of \( G \). A pc presentation for a group \( G \) of order \( p^m \) is called consistent if the presentation has \( m \) pc generators.

If the finite group \( G \) has exponent \( p \), then a pc presentation for \( G \) has the form
\[
\langle a_1, \ldots, a_n | a_i^p = 1, 1 \leq i \leq n, [a_k, a_j] = v_{jk}, 1 \leq j < k \leq n \rangle,
\]
where \( v_{jk} \) is a word in the elements \( a_{k+1}, \ldots, a_n \).

If the \( p \)-group \( G \) has a consistent pc presentation, then every group element can be written uniquely as a word in the normal form \( a_1^{e_1} a_2^{e_2} \ldots a_n^{e_n} \), where \( 0 \leq e_i < p \) for \( 1 \leq i \leq n \) ([53]). Additionally, every group of order \( p^m \), where \( p \) is a prime number, has a consistent pc presentation([54]).
Definition 2.2. Let \( G \) be a group and \( p \) a prime number. The \( p \)-central series
\[ P_0(G) = G \supseteq P_1(G) \supseteq \cdots \supseteq P_{c+1}(G) = 1 \]
of \( G \) is defined by \( P_0(G) = G \) and
\[ P_i(G) = [G,P_{i-1}(G)](P_{i-1}(G))^p \]
for \( i \geq 1 \). Because \( G \) is a finite \( p \)-group, the \( p \)-central series of \( G \) eventually ends in the trivial group. The \( p \)-class of \( G \) is the smallest integer \( c \) such that \( P_{c+1}(G) = 1 \).

We note that if \( G \) is a group of exponent \( p \), then the \( p \)-central series of \( G \) is the same as the lower central series of \( G \).

Definition 2.3. Let \( G \) be a finite \( p \)-group and let \( \{a_1, \ldots, a_n\} \) be a set of \( p \) generators for a consistent \( p \)c presentation of \( G \). Suppose the composition series
\[ \langle 1 \rangle < \langle a_n \rangle < \langle a_n, a_{n-1} \rangle < \cdots < G \]
refines the \( p \)-central series for \( G \). We define a weight function \( w \) on the \( p \)c generators \( \{a_1, \ldots, a_n\} \) by \( w(a_i) = j \) if \( a_i \) is contained in \( P_{j-1}(G) \) but not in \( P_j(G) \). In this case the \( p \)c presentation is called a weighted \( p \)c presentation.

In the following, we will at times use the term \( p \)-weight for the weight function on a \( p \)c-presentation.

Sims [55, p 447] shows that every finite \( p \)-group has a weighted \( p \)c presentation. He also gives the following properties of a weighted \( p \)c presentation:

1. \( w(a_i) \leq w(a_{i+1}) \) for \( 1 \leq i < n \).
2. For \( 1 \leq i \leq n \) there is a power relation \( a_i^p = v_{ii} \) where the \( p \)c generators
   that occur in \( v_{ii} \) all have weight at least \( w(a_i) + 1 \).
3. For the relation \([a_i, a_j] = v_{ij} \), the word \( v_{ij} \) contains only generators of
   weight at least \( w(a_i) + w(a_j) \).
4. If \( a_k \) is a \( p \)c generator with \( w(a_k) > 1 \), then one of the following holds:
   a. There are indices \( i \) and \( j \) such that \( a_k = [a_i, a_j] \) with \( w(a_i) = w(a_k) - 1 \)
      and \( w(a_j) = 1 \).
   b. There exists an index \( i \) such that \( a_k = a_i^p \) with \( w(a_i) = w(a_k) - 1 \).
If $G$ has exponent $p$, then these properties show that any generator $a_k$ of weight greater than one is the commutator of a generator $a_j$ of weight $w(a_k) - 1$ and a generator $a_i$ of weight one. Also, the weight of a pc generator is the same as its commutator weight.

We also see that a weighted pc presentation must have at least one generator of each weight up to and including the $p$-class.

**Definition 2.4.** Let $G$ be a finite $p$-group and let $\{a_1, \ldots, a_n\}$ be a set of pc generators for a weighted pc presentation of $G$. Let $d$ be the $p$-class of $G$. In this dissertation the pc structure of $G$ is a finite sequence of integers $b_1, \ldots, b_d$, where $b_i$ is the number of pc generators of $G$ that are of weight $i$ or less.

## 2. Computer Background

The computer package GAP (short for Groups, Algorithms, and Programming) [56] is used for many calculations in Chapter 3. In particular, the package ANUPQ [57] is used for computing quotients. In the construction of various examples the function $Pq$ from the package ANUPQ is frequently used. To use this function to explore a group $G$, we input a free group on a certain number of generators. We also give a list of relators that are satisfied in $G$ and a prime $p$. We optionally can provide a bound $b$ that specifies the last term of the $p$-central series to calculate. The function $Pq$ then determines a set of factor groups of the subgroups of the $p$-central series for $G$. It stops calculating the factor groups at $p$-class $b$, or else stops when the $p$-central series becomes trivial, whichever comes first. We may also impose exponent laws on $G$ via an option we give the $Pq$ function, and impose other laws (usually the $n$-Engel law for a small integer $n$) if needed. As GAP and Pq are calculating, GAP gives a list of the sum of the number of pc generators of the factor groups calculated so far. In fact, we read the pc structure of $G$ directly from these numbers.
Many of the groups we investigate have exponent $p$, in which case the $p$-central series is the same as the lower central series. In this case the $p$-weights of the pc generators are their standard commutator weights. Also, the $p$-class of a group of exponent $p$ is the same as the nilpotence class of the group.

In Chapter 5, we also use the computer software Maple \cite{58} for number theory calculations.

3. Building a Presentation

For this section, fix a prime number $p$. We show how to construct a pc presentation for a finite $p$-group $G$ using GAP. Choose a number $m$ of group generators for $G$. Then the number of pc generators of $G$ of weight 1 is also $m$. We also choose a $p$-class $c$. Usually we also pick an exponent $e$, which is a power of $p$. We now pick an ordering for the weight 1 generators and label them $a_1$ to $a_m$. If we are building the group in GAP, then we fulfill this step by starting out with a free group on $m$ generators. For the rest of this section, we will construct examples using $m = 2$, $p = 5$, $e = 5$, and $c = 3$. The following GAP code will give $G$:

```gap
RequirePackage("anupq");
f:=FreeGroup(2);
g:=Pq(f: Prime:=5, Exponent:=5, ClassBound:=3);
```

Now we decide which weight 2 commutators we want as pc generators in $G$, and what power generators we want. The power generators will all be $p$ powers of weight 1 generators. Because $G$ has exponent $p$, we need only pick weight 2 commutators since $a_i^p$ is always 1. In GAP, it is easier to pick which of the possible generators we wish to be trivial. This is done by adding weight 2 commutators to the list of relators which we give GAP. For example,

```gap
RequirePackage("anupq");
f:=FreeGroup(2);
rels:=["[f1,f2"]];
g:=Pq(f: Prime:=5, Exponent:=5, Relators:=rels, ClassBound:=3);
```
This calculation gives the group $C_5 \times C_5$. We have made the only possible weight 2 pc generator $[a_1, a_2]$ trivial, so our pc presentation has 5-class 1. For this reason, in the following we will assume that $G$ has no relators of weight 2 so that we will obtain a weight 2 pc generator. If the free group had more than 2 generators, we could have a weight 2 relator and still have other weight 2 pc generators.

We continue by working with the possible weight 3 pc generators. In the previous step, we decided that there is one weight 2 pc generator, $[a_1, a_2]$. The possible weight 3 pc generators are $[a_1, a_2, a_1]$ and $[a_1, a_2, a_2]$. If we had not specified that we were working with a group of exponent 5, then we would also need to consider the 5th power of each weight 2 generator, such as $[a_1, a_2]^5$. For our example, we decide that $[a_1, a_2, a_1]$ is a pc generator and that $[a_1, a_2, a_2]$ is trivial. When we are using GAP, it is again easier to indicate which possible weight 3 simple commutators in the group generators we wish to be trivial. Note that not all weight 3 simple commutators are necessarily either pc generators or relators. We can construct the group $G$ in GAP by the code:

```gap
RequirePackage("anupq");
f:=FreeGroup(2);
rels:="[f1,f2,f2]";
g:=Pq(f: Prime:=5, Exponent:=5, Relators:=rels, ClassBound:=3);
```

We continue this process until we reach the $p$-class that we decided on at first. In our example, we have reached our 5-class of 3, so we stop.

In practice, we usually want to create a group that has additional properties. It is common to want to add a law to a group. For example, the following GAP code includes the 3-Engel law.

```gap
RequirePackage("anupq");
f:=FreeGroup(3);
rels:="[f1,f2,f2]";
g:=Pq(f: Prime:=5, Exponent:=5, ClassBound:=5, Relators:=rels, Identities:=[function(a,b) return LeftNormedComm([a,b,b,b]); end]);
```

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Sometimes we want to construct a group that does not have a particular property $P$. If, for instance, we want a group where the nilpotence class of the normal closure of the first generator is greater than 2, then we add relators as shown above, testing the nilpotence class of the normal closure of the first generator after each relator is added. If the group which has the questioned relator has property $P$, then we know we do not want that relator. This is the method used to generate many of the examples in Chapter 3.
Chapter 3

Examples Where $\mathfrak{e}_n$ Does Not Imply $\mathcal{L}(\mathfrak{N}_{n-1})$

1. Gupta and Levin Counterexample

In this section we investigate the details of an example by Gupta and Levin [3] that shows that condition $\mathcal{L}(\mathfrak{N}_3)$ is not implied by condition $\mathfrak{e}_4$.

1.1. Example of Gupta and Levin. In this subsection, we construct in detail the first published example, by Gupta and Levin, of a 4-Engel group that has an element whose normal closure in the group is not nilpotent of class 3. Our explanation here contains many more details than the original explanation in [3].

We will need the following lemmas:

**Lemma 3.1** (Chapter 5 of [59]). Let $A$ be an associative algebra and $L$ its related Lie algebra. Let $G$ be a group whose generators are of the form $1 + a$, where $a$ is in $L$. Then $[1 + a_1, 1 + a_2, \ldots, 1 + a_n] = 1 + (a_1, a_2, \ldots, a_n) + t$, where $t$ is a sum of Lie commutators of weight greater than $n$ in $L$.

**Lemma 3.2** (Lemma 4.1 in [3]). Let $p$ be an odd prime. Let $G$ be a group of exponent $p$ that is nilpotent of class at most $p + 1$. Then $G$ is $(p - 1)$-Engel if and only if $G$ satisfies the law $[x, y, x, p-2y] = 1$.

In Lemma 3.4, we will use the Diamond Lemma for algebras [60]. We follow the exposition of Appendix I.11, pages 97-102, from [61].

**Definition 3.1.** Let $F$ be a free algebra on a set of generators $\langle X \rangle$ and $A$ an algebra that is a quotient of $F$. Then we can write $A = F/\langle w_\sigma - f_\sigma \mid \sigma \in \Sigma \rangle$, where $f_\sigma \in F$ and $w_\sigma$ is a word in $X$. We call $S = \{(w_\sigma, f_\sigma) \mid \sigma \in \Sigma \}$ a reduction system, and each $(w_\sigma, f_\sigma)$ in $S$ we call a reduction. An element of $F$ is called irreducible if
no reduction from $S$ alters it. An ambiguity is a word in $X$ to which we could apply more than one reduction. A reduction system $S$ and an ordering $\leq$ on the set of words on $X$ are compatible when for every $\sigma \in \Sigma$, the element $f_\sigma$ is a linear combination of words $w < w_\sigma$.

**Lemma 3.3 (The Diamond Lemma).** (see [60]) Let $F = k\langle X \rangle$ be a free algebra on a set $X$ and $W$ the free monoid on $X$. Let $S = \{(w_\sigma, f_\sigma) \mid \sigma \in \Sigma\}$ be a reduction system of $F$, and $\leq$ a semigroup ordering on $W$ which is compatible with $S$ and satisfies the Descending Chain Condition. If all overlap and inclusion ambiguities are resolvable, then the cosets $\bar{w}$, for irreducible words $w \in W$, form a basis for the factor algebra $F/\langle w_\sigma - f_\sigma \mid \sigma \in \Sigma\rangle$.

Let $R$ be the free associative algebra in three non-commuting indeterminates, $x_0, x_1$ and $x_2$, over the field $\mathbb{Z}_5$. Construct an ideal in the following manner:

Set $S_1 = \{\text{monomials of degree 7 in } R\}$, $S_2 = \{\text{monomials in } R \text{ that are of degree 2 in } x_1 \text{ or of degree 2 in } x_2\}$, $S_3 = \{x_1x_2, x_2x_1, x_0^3, x_0^2x_1, x_0^2x_2\}$, and $S_4 = \{x_1x_0x_2 + x_2x_0x_1\}$. Let $I$ be the ideal in $R$ generated by $S_1$, $S_2$, $S_3$, and $S_4$. Set $A = R/I$.

**Lemma 3.4.** (Gupta and Levin [3]) Let $H$ be the set of all products of powers of the elements $1 + x_0$, $1 + x_1$, and $1 + x_2$ in $A$ (including the empty product, which is 1 in $A$). Then $H$ is a multiplicative group that has exponent 5, has order at most $5^{20}$, is nilpotent of class exactly 6, is 4-Engel, and contains an element whose normal closure in $H$ is not nilpotent of class 3.

**Proof.** We begin with an examination of the monomials in $A$ using the Diamond Lemma (Lemma 3.3). We can see that $A$ is determined by the following reduction system $S$: for each element $w$ of $S_1$, $S_2$, and $S_3$ we have a reduction $(w, 0)$. The only other reduction needed is $(x_2x_0x_1, -x_1x_0x_2)$. In order to apply the Diamond Lemma, we must define a partial ordering on the free monoid $\langle x_0, x_1, x_2 \rangle$ that has the descending chain condition and is compatible with $S$. We consider the
length-lexicographic order, first defining \( x_0 < x_1 < x_2 \). In this ordering, for words \( w \) and \( v \), we define \( w < v \) if the length of \( w \) is less than the length of \( v \), or else if \( w \) and \( v \) have the same length and \( w \) precedes \( v \) in the lexicographic order. This ordering is consistent with every \((w, 0)\) reduction because \( 0 < w \). It also is consistent with \((x_2 x_0 x_1, -x_1 x_0 x_2)\) because \( x_1 x_0 x_2 < x_2 x_0 x_1 \). This ordering also satisfies the descending chain condition. Thus, we have an ordering required by the Diamond Lemma. Now, we need to show that all ambiguities in \( S \) are resolvable. We do not need to be concerned about reductions \((w, 0)\) and \((v, 0)\), because the use of either reduction first will still result in \( 0 \). Thus, all we need to consider is an ambiguity between reductions \((w, 0)\) and \((x_2 x_0 x_1, -x_1 x_0 x_2)\). If \( w \in S_1 \), there is no problem because the reduction \((x_2 x_0 x_1, -x_1 x_0 x_2)\) does not alter the length of words. Similarly, \((x_2 x_0 x_1, -x_1 x_0 x_2)\) does not alter the weight of a word in \( x_1 \) or \( x_2 \), so ambiguities with \( w \in S_2 \) are also resolvable. The only remaining case is when \( w \in S_3 \). We see that \( w \) cannot be \( x_0^3 \), \( x_2 x_1 \), or \( x_0^2 x_1 \) because there is no possible overlap or containment with \( x_2 x_0 x_1 \). If \( w = x_1 x_2 \), the only possible minimal overlaps are the monomials \( x_1 x_2 x_0 x_1 \) and \( x_2 x_0 x_1 x_2 \). If we first apply \((x_2 x_0 x_1, -x_1 x_0 x_2)\), we get \(-x_1^2 x_0 x_2\) and \(-x_1 x_0 x_2^2\) and we then apply a reduction from \( S_2 \) and get \( 0 \). If we apply the reduction from \( S_2 \) first, the result in either case is \( 0 \), so the ambiguity is resolvable. If \( w = x_0^2 x_2 \), the only possible minimal overlap is the monomial \( x_0^2 x_2 x_0 x_1 \). If we first apply \((x_2 x_0 x_1, -x_1 x_0 x_2)\), then we get \(-x_0^2 x_1 x_0 x_2\) and apply the reduction \((x_0^2 x_1, 0)\) to get \( 0 \). If we apply the reduction \((x_0^2 x_1, 0)\) first, we again get \( 0 \). Thus, by the Diamond Lemma, the irreducible words form a basis for \( A \). In Table 1 we list all nontrivial irreducible words, namely, the words to which we can not apply any reduction. Thus, the monomials in Table 1, together with \( 1 \), provide a basis for \( A \).

We treat \( A \) as consisting of all linear combinations over \( \mathbb{Z}_5 \) of monomials from Table 1. Then \( H \) contains only elements of the form \( 1 + y \), where \( y \) is a linear combination of monomials from Table 1.
Table 1: Irreducible Monomials in $A$

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_0x_1$</th>
<th>$x_0x_2$</th>
<th>$x_0^2$</th>
<th>$x_1x_0$</th>
<th>$x_2x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0x_1x_0$</td>
<td>$x_0x_2x_0$</td>
<td>$x_1x_0x_2$</td>
<td>$x_0x_1^2$</td>
<td>$x_0^2x_0$</td>
<td>$x_1^2$</td>
<td>$x_0x_2^2$</td>
<td>$x_1x_2^2$</td>
</tr>
<tr>
<td>$x_0x_1x_0^2x_0$</td>
<td>$x_1x_0x_2x_0$</td>
<td>$x_0x_1x_2^2$</td>
<td>$x_0^2x_2^2$</td>
<td>$x_1x_2^2$</td>
<td>$x_2x_0^2$</td>
<td>$x_0x_2^2x_0$</td>
<td></td>
</tr>
</tbody>
</table>

We verify the following:

1. The set $H$ is closed under multiplication
2. $H$ contains an identity (which also shows that $H$ is nonempty)
3. Each element in $H$ has an inverse in $H$
4. $H$ is of exponent 5
5. $H$ is of order $\leq 5^{20}$
6. $H$ is a 4-Engel group
7. $H$ is nilpotent of class 6
8. There is at least one element of $H$ whose normal closure in $H$ is not nilpotent of class 3.

(1) and (2) follow immediately from the definition of $H$.

We now show that all elements of $H$ are of exponent 5. Let $m$ be an element of $H$. Then $m = 1 + n$, where $n$ is a linear combination of monomials from Table 1. The coefficients in $m$ are in $\mathbb{Z}_5$, so $m^5 = 1 + n^5$. However, every term of $n^5$ is a product of 5 monomials, and monomials of degree greater than 6 are trivial in $A$, so we may assume that $n$ contains only monomials of degree 1 or 2. Let $n = a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_0 + a_7x_2x_0$. In the following calculations, we gather together and label with uppercase letters terms of weight large enough to become trivial by the end of the calculation.

\[
\begin{align*}
n^2 &= (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_0 + a_7x_2x_0)^2 \\
&= a_0^2x_0^2 + a_0a_1x_0x_1 + a_0a_2x_0x_2 + a_0a_1x_1x_0 + a_0a_2x_2x_0 + (a_0a_6 + a_4a_0)x_0x_1x_0 \\
&\quad + (a_0a_7 + a_5a_0)x_0x_2x_0 + (a_1a_3 + a_6a_0)x_1x_0^2 + (a_2a_3 + a_7a_0)x_2x_0^2 \\
&\quad + (a_1a_5 - a_2a_4 + a_6a_2 - a_7a_1)x_1x_0x_2 + N,
\end{align*}
\]
where \( N \) is a sum of monomials of degree 4 or more. Further,

\[
n^4 = (a_0^2 x_0^2 + a_0 a_1 x_0 x_1 + a_0 a_2 x_0 x_2 + a_0 a_1 x_1 x_0 + a_0 a_2 x_2 x_0 + (a_0 a_6 + a_4 a_0) x_0 x_1 x_0 \\
+ (a_0 a_7 + a_5 a_0) x_0 x_2 x_0 + (a_1 a_3 + a_6 a_0) x_1 x_0^2 + (a_2 a_3 + a_7 a_0) x_2 x_0^2) \\
+ (a_1 a_5 - a_2 a_4 + a_6 a_2 - a_7 a_1) x_1 x_0 x_2 + N)^2 \\
= a_0^3 a_1 x_0 x_1 x_0^2 + a_0^3 a_2 x_0 x_2 x_0^2 + (a_0^2 a_1 a_5 - a_0^2 a_2 a_4) x_1 x_0 x_2 x_0^2 + M,
\]

where \( M \) is a sum of monomials of degree 6 or more. Finally,

\[
n^5 = (a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_0^2 + a_4 x_0 x_1 + a_5 x_0 x_2 + a_6 x_1 x_0 + a_7 x_2 x_0) \\
\cdot (a_0^3 a_1 x_0 x_1 x_0^2 + a_0^3 a_2 x_0 x_2 x_0^2 + (a_0^2 a_1 a_5 - a_0^2 a_2 a_4) x_1 x_0 x_2 x_0^2 + M) \\
= 0.
\]

We conclude that \( H \) is of exponent 5, which shows (4).

(3) follows since every nonidentity element of \( H \) is of order 5.

(5) holds because there are \( 5^{20} \) linear combinations over \( \mathbb{Z}_5 \) of the monomials in Table 1.

Next we prove (7). Let \( a, b, c, d, e, f, \) and \( g \) be elements of \( A \), so that \( 1 + a, 1 + b, 1 + c, 1 + d, 1 + e, 1 + f, \) and \( 1 + g \) are elements of \( H \). By Lemma 3.1, \( [1 + a, 1 + b, 1 + c, 1 + d, 1 + e, 1 + f, 1 + g] = 1 + (a, b, c, d, e, f, g) + t \), where \( t \) is a sum of Lie commutators of weight 8 or more. However, in \( A \) only terms of weight 6 or less are nontrivial, so \( t \) is trivial. Similarly, \( (a, b, c, d, e, f, g) \) has weight 7 and so is trivial. Thus a commutator of weight 7 or more in \( H \) is trivial. Hence the nilpotence class of \( H \) is less than or equal to 6. To show that \( H \) has nilpotence class 6, we consider the commutator \([1 + a, 1 + b, 1 + b, 1 + b, 1 + c, 1 + b] \). By Lemma 3.1, \([1 + a, 1 + b, 1 + b, 1 + b, 1 + c, 1 + b] = 1 + (a, b, b, c, b) \). Now we let \( a = x_1, b = x_0, \) and \( c = x_2 \). Then
Because \([1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0, 1 + x_2, 1 + x_0] \in H\) is not trivial, the nilpotence class of \(H\) is exactly 6.

Now we can prove (6). To show that \(H\) is 4-Engel by Lemma 3.2, we show that 
\([x,y,x,y,y] = 1\) for all \(x,y \in H\). Let \(x = 1 + j\) and \(y = 1 + k\) where \(i\) and \(j\) are in \(A\). By Lemma 3.1, it suffices to show that \((j,k,j,k,k,k) = 0\).

We may assume without loss of generality that \(j\) and \(k\) contain only monomials of degree 1, because any greater degree terms in \((j,k,j,k,k,k)\) would give terms that are in \(S_1\) and so are trivial in \(H\). Thus \(j = j_0 x_0 + j_1 x_1 + j_2 x_2\) and \(k = k_0 x_0 + k_1 x_1 + k_2 x_2\), where \(j_0, j_1, j_2, k_0, k_1,\) and \(k_2\) are in \(\mathbb{Z}_5\).

Now,

\[
jk = (j_0 x_0 + j_1 x_1 + j_2 x_2)(k_0 x_0 + k_1 x_1 + k_2 x_2)
= j_0 k_0 x_0^2 + j_0 k_1 x_0 x_1 + j_0 k_2 x_0 x_2
+ j_1 k_0 x_1 x_0 + j_1 k_1 x_1^2 + j_1 k_2 x_1 x_2
+ j_2 k_0 x_2 x_0 + j_2 k_1 x_2 x_1 + j_2 k_2 x_2^2.
\]

Because \(x_1^2, x_2^2, x_1 x_2,\) and \(x_2 x_1\) are all in \(I\), we have
Further, \( \ell \)

In the following, we use the notations

We also abbreviate \( j \)

By symmetry,

Thus

\[
(j, k) = jk - k
\]

\[
= (j_0k_0x_0^2 + j_0k_1x_0x_1 + j_0k_2x_0x_2 + j_1k_0x_1x_0 + j_2k_0x_2x_0)
\]

\[
- (j_0k_0x_0^2 + j_1k_0x_0x_1 + j_2k_0x_0x_2 + j_0k_1x_1x_0 + j_0k_2x_2x_0)
\]

\[
= (j_0k_1 - j_1k_0)x_0x_1 + (j_0k_2 - j_2k_0)x_0x_2
\]

\[
- (j_0k_1 - j_1k_0)x_1x_0 - (j_0k_2 - j_2k_0)x_2x_0.
\]

In the following, we use the notations \( \ell_1 \) for \( j_0k_1 - j_1k_0 \) and \( \ell_2 \) for \( j_0k_2 - j_2k_0 \). Then

\[
(j, k) = \ell_1x_0x_1 + \ell_2x_0x_2 - \ell_1x_1x_0 - \ell_2x_2x_0.
\]

We also abbreviate \( j_1k_2 - j_2k_1 \) by \( \ell_3 \). Then \( \ell_2k_1 - \ell_1k_2 = \ell_3k_0 \). Continuing,

\[
(j, k) = (j, k)j - j(j, k)
\]

\[
= (\ell_1x_0x_1 + \ell_2x_0x_2 - \ell_1x_1x_0 - \ell_2x_2x_0)(j_0x_0 + j_1x_1 + j_2x_2)
\]

\[
- (j_0x_0 + j_1x_1 + j_2x_2)(\ell_1x_0x_1 + \ell_2x_0x_2 - \ell_1x_1x_0 - \ell_2x_2x_0)
\]

\[
= \ell_1j_0x_0x_1x_0 + \ell_2j_0x_0x_2x_0 - \ell_1j_0x_1x_0^2
\]

\[
- \ell_1j_2x_1x_0x_2 - \ell_2j_0x_2x_0^2 - \ell_2j_1x_2x_0x_1
\]

\[
+ j_0\ell_1x_0x_1x_0 + j_0\ell_2x_0x_2x_0 - j_1\ell_2x_1x_0x_2 - j_2\ell_1x_2x_0x_1
\]

\[
= 2\ell_1j_0x_0x_1x_0 + 2\ell_2j_0x_0x_2x_0 - \ell_1j_0x_1x_0^2 - \ell_2j_0x_2x_0^2
\]

\[
= j_0 \left( 2\ell_1x_0x_1x_0 + 2\ell_2x_0x_2x_0 - \ell_1x_0^2 - \ell_2x_0^2 \right).
\]

Further,
\((j, k, j, k) = (j, k, j)k - k(j, k, j)\)
\[
\begin{align*}
(j, k, j, k) &= j_0((2l_1x_0x_1x_0 + 2l_2x_0x_2x_0 - l_1x_1x_0^2 - l_2x_0x_2^2)(k_0x_0 + k_1x_1 + k_2x_2) \\
&\quad - (k_0x_0 + k_1x_1 + k_2x_2)(2l_1x_0x_1x_0 + 2l_2x_0x_2x_0 - l_1x_1x_0^2 - l_2x_0x_2^2)) \\
&= j_0(2l_1k_0x_0x_1x_0^2 + 2l_2k_0x_0x_2x_0^2 + 2l_2k_1x_0x_2x_0x_1 + 2l_1k_2x_0x_1x_0x_2 \\
&\quad + k_0l_1x_0x_1x_0^2 + k_0l_2x_0x_2x_0^2 - 2k_1l_2x_0x_2x_0x_0 - 2k_2l_1x_0x_2x_0x_0) \\
&= j_0(3l_1k_0x_0x_1x_0^2 + 3l_2k_0x_0x_2x_0^2 \\
&\quad - 2(l_2k_1 - l_1k_2)x_0x_1x_0x_2 - 2(l_2k_1 - l_1k_2)x_1x_0x_0x_2) \\
&= j_0(3l_1k_0x_0x_1x_0^2 + 3l_2k_0x_0x_2x_0^2 - 2l_3k_0x_1x_0x_2 - 2l_3k_0x_1x_0x_2x_0) \\
&= 3j_0k_0(l_1x_0x_1x_0^2 + l_2x_0x_2x_0^2 + l_3x_0x_1x_0x_2 + l_3x_1x_0x_2x_0).
\end{align*}
\] (11)

Thus,
\[
(j, k, j, k, k) = (j, k, j, k)k - k(j, k, j, k)
\begin{align*}
&= 3j_0k_0((l_1x_0x_1x_0^2 + l_2x_0x_2x_0^2 + l_3x_0x_1x_0x_2 \\
&\quad + l_3x_0x_0x_2x_0)(k_0x_0 + k_1x_1 + k_2x_2) \\
&\quad - (k_0x_0 + k_1x_1 + k_2x_2)((l_1x_0x_1x_0^2 \\
&\quad + l_2x_0x_2x_0^2 + l_3x_0x_1x_0x_2 + l_3x_1x_0x_2x_0)) \\
&= 3j_0k_0(\ell_3k_0x_0x_1x_0x_2x_0 + \ell_3k_0x_1x_0x_2x_0 \\
&\quad - k_0\ell_3x_0x_1x_0x_2x_0 - k_1\ell_2x_1x_0x_2x_0^2 - k_2\ell_1x_2x_0x_1x_0^2) \\
&= 3j_0k_0(\ell_3k_0x_1x_0x_2x_0^2 - k_1\ell_2x_1x_0x_2x_0^2 - k_2\ell_1x_2x_0x_1x_0^2) \\
&= 3j_0k_0(\ell_3k_0 - k_1\ell_2 + k_2\ell_1)x_0x_2x_0^2.
\end{align*}
\] (12)

Because \(\ell_2k_1 - \ell_1k_2 = \ell_3k_0\), it follows that \((j, k, j, k, k) = 0\), which completes the proof of (6). It should be noted that because of restrictions in the statement of Lemma 3.1, we have not proved that \([1 + j, 1 + k, 1 + j, 1 + k, 1 + k]\) is trivial in \(H\), only that \([1 + j, 1 + k, 1 + j, 1 + k, 1 + k]\) is \(1 + s\), where \(s\) is a sum of weight six Lie commutators in \(A\).

Finally to prove (8), we show that \([1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0, 1 + x_2, 1 + x_0]\) is in \(\gamma_4((1 + x_0)^{17})\). We showed in the proof of (7) that
[1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0, 1 + x_2, 1 + x_0] is not trivial, so it will follow that
\( \gamma_4((1 + x_0)^H) \) is not trivial, which will complete the proof.

We first notice that \([1 + x_1, 1 + x_0] = ((1 + x_0)^{-1})^{(1 + x_1)}(1 + x_0)\), so \([1 + x_1, 1 + x_0]\)
is in \((1 + x_0)^H\). Then because \((1 + x_0)\) is also in \((1 + x_0)^H\), we know that
\([1 + x_1, 1 + x_0, 1 + x_0] \) lies in \( \gamma_2((1 + x_0)^H) \). Similarly, \([1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0] \)
is contained in \( \gamma_3((1 + x_0)^H) \). Thus \([1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0, 1 + x_2] =
[1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0]^{-1}([1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0])^{(1 + x_2)}\), so
\([1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0, 1 + x_2] \) is an element of \( \gamma_3((1 + x_0)^H) \) and therefore
\([1 + x_1, 1 + x_0, 1 + x_0, 1 + x_0, 1 + x_2, 1 + x_0] \) is contained in \( \gamma_4((1 + x_0)^H) \).

Hence the element \((1 + x_0)\) of \( H \) has normal closure that is not nilpotent of
class 3. Thus all the claimed properties of \( H \) are verified. □

Lemma 3.4 immediately proves the major result of this section.

**Theorem 3.5.** (Gupta and Levin, [3]) There exists a 4-Engel 5-group \( H \) with
an element \( a \) whose normal closure \( a^H \) is not nilpotent of class 3.

**1.2. Relators.** In this subsection, we show that certain commutators in the
Gupta and Levin example, which we call \( H \), are trivial. First, we need a lemma that
gives the inverses of some of the elements of \( H \).

**Lemma 3.6.** In \( H \), if \( w \) is a sum of words in \( x_0, x_1, \) and \( x_2, \) and either every
word in \( w \) is of degree at least one in \( x_1 \) or every word in \( w \) is of degree at least one
in \( x_2, \) then \((1 + w)^{-1} = 1 - w.\) In particular, \((1 + x_1)^{-1} = 1 - x_1 \) and
\((1 + x_2)^{-1} = 1 - x_2.\) Also, \((1 + x_0)^{-1} = 1 - x_0 + x_0^2.\)

**Proof.** We note that \((1 + w)(1 - w) = 1 - w^2.\) Because \( w^2 \) contains only words
of degree at least 2 in either \( x_1 \) or \( x_2, \) we see \( w^2 \) is contained in \( S_2 \) and so is 0. This
proves that \((1 + w)(1 - w) = 1, \) so \((1 + w)^{-1} = 1 - w.\) Also,
\((1 + x_0)(1 - x_0 + x_0^2) = (1 - x_0 + x_0^2 + x_0 - x_0^2 + x_0^3) = 1, \) which completes the proof. □
Lemma 3.7. In $H$ the commutators $[1 + x_1, 1 + x_2]$, $[1 + x_0, 1 + x_1, 1 + x_1]$, $[1 + x_0, 1 + x_2, 1 + x_2]$, and $[1 + x_0, 1 + x_1, 1 + x_2, 1 + x_0, 1 + x_0]$ are all trivial. Further, $[1 + x_0, 1 + x_1, 1 + x_2] = [1 + x_0, 1 + x_2, 1 + x_1]^{-1}$.

Proof. To prove this lemma, we calculate the commutators $[1 + x_1, 1 + x_2]$, $[1 + x_0, 1 + x_1, 1 + x_1]$, $[1 + x_0, 1 + x_2, 1 + x_2]$, and $[1 + x_0, 1 + x_1, 1 + x_2, 1 + x_0, 1 + x_0]$. First, using Lemma 3.6,

$$[1 + x_1, 1 + x_2] = (1 + x_1)^{-1}(1 + x_2)^{-1}(1 + x_1)(1 + x_2)$$
$$= (1 - x_1)(1 - x_2)(1 + x_1 + x_2)$$
$$= (1 - x_1)(1 + x_2 - x_2)$$
$$= (1 - x_1)(1 + x_1)$$
$$= 1. \ (13)$$

Also, again using 3.6,

$$[1 + x_0, 1 + x_1] = (1 + x_0)^{-1}(1 + x_1)^{-1}(1 + x_0)(1 + x_1)$$
$$= (1 - x_0 + x_0^2)(1 - x_1)(1 + x_0 + x_1 + x_0x_1)$$
$$= (1 - x_0 + x_0^2)(1 + x_0 + x_1 + x_0x_1 - x_1 - x_1x_0)$$
$$= (1 - x_0 + x_0^2)(1 + x_0 + x_0x_1 - x_1x_0)$$
$$= 1 + x_0 + x_0x_1 - x_1x_0 - x_0 - x_0^2 + x_0x_1x_0 + x_0^2$$
$$= 1 + x_0x_1 - x_1x_0 + x_0x_1x_0. \ (14)$$

It follows that

$$[1 + x_0, 1 + x_1, 1 + x_1] = (1 + x_0x_1 - x_1x_0 + x_0x_1x_0)^{-1}(1 + x_1)^{-1}$$
$$\cdot (1 + x_0x_1 - x_1x_0 + x_0x_1x_0)(1 + x_1)$$
$$= (1 - x_0x_1 + x_1x_0 - x_0x_1x_0)(1 - x_1)$$
$$\cdot (1 + x_0x_1 - x_1x_0 + x_0x_1x_0 + x_1 - x_1)$$
$$= (1 - x_0x_1 + x_1x_0 - x_0x_1x_0)$$
$$\cdot (1 + x_0x_1 - x_1x_0 + x_0x_1x_0 + x_1 - x_1)$$
$$\cdot (1 + x_0x_1 - x_1x_0 + x_0x_1x_0)$$
$$\cdot (1 + x_0x_1 - x_1x_0 + x_0x_1x_0)$$
$$= 1. \ (15)$$
By symmetry, \([1 + x_0, 1 + x_2, 1 + x_2] = 1\). In addition, using (14) again, along with Lemma 3.6,

\[
[1 + x_0, 1 + x_1, 1 + x_2] = (1 + x_0 x_1 - x_1 x_0 + x_0 x_1 x_0)^{-1}(1 + x_2)^{-1} \\
\cdot (1 + x_0 x_1 - x_1 x_0 + x_0 x_1 x_0)(1 + x_2) \\
= (1 - x_0 x_1 + x_1 x_0 - x_0 x_1 x_0) (1 - x_2) (1 + x_0 x_1 \\
- x_1 x_0 + x_0 x_1 x_0 + x_2 - x_1 x_0 x_2 + x_0 x_1 x_0 x_2) \\
= (1 - x_0 x_1 + x_1 x_0 - x_0 x_1 x_0) (1 + x_0 x_1 \\
- x_1 x_0 + x_0 x_1 x_0 + x_2 - x_1 x_0 x_2 \\
+ x_0 x_1 x_0 x_2 - x_2 x_0 x_1 x_0) \\
= (1 - x_0 x_1 + x_1 x_0 - x_0 x_1 x_0) (1 + x_0 x_1 - x_1 x_0 \\
+ x_0 x_1 x_0 + x_0 x_1 x_0 x_2 - x_2 x_0 x_1 x_0) \\
= 1 + x_0 x_1 x_0 x_2 - x_2 x_0 x_1 x_0.
\]

By symmetry, \([1 + x_0, 1 + x_2, 1 + x_1] = 1 + x_0 x_2 x_0 x_1 - x_1 x_0 x_2 x_0 = [1 + x_0, 1 + x_1, 1 + x_2]^{-1}\).

In the following, we use \(z\) to denote \([1 + x_0, 1 + x_1, 1 + x_2]\). Using (16) and Lemma 3.6,

\[
[z, 1 + x_0] = (1 + x_0 x_1 x_0 x_2 - x_2 x_0 x_1 x_0)^{-1}(1 + x_0)^{-1} \\
\cdot (1 + x_0 x_1 x_0 x_2 - x_2 x_0 x_1 x_0)(1 + x_0) \\
= (1 - x_0 x_1 x_0 x_2 + x_2 x_0 x_1 x_0) (1 - x_0 + x_0^2) \\
\cdot (1 + x_0 x_1 x_0 x_2 - x_2 x_0 x_1 x_0 + x_0 + x_0 x_1 x_0 x_2 x_0 - x_2 x_0 x_1 x_0^2) \\
= (1 - x_0 x_1 x_0 x_2 + x_2 x_0 x_1 x_0) (1 + x_0 x_1 x_0 x_2 \\
- x_2 x_0 x_1 x_0 + x_0 + x_0 x_1 x_0 x_2 x_0 - x_2 x_0 x_1 x_0^2 \\
- x_0 + x_0 x_2 x_0 x_1 x_0 - x_0^2 + x_0 x_2 x_0 x_1 x_0^2 + x_0^2) \\
= (1 - x_0 x_1 x_0 x_2 + x_2 x_0 x_1 x_0) (1 + x_0 x_1 x_0 x_2 \\
- x_2 x_0 x_1 x_0 - x_2 x_0 x_1 x_0^2 + x_0 x_2 x_0 x_1 x_0^2) \\
= 1 - x_2 x_0 x_1 x_0^2 + x_0 x_2 x_0 x_1 x_0^2.
\]

Further,

\[
[z, 1 + x_0, 1 + x_0] = (1 - x_2 x_0 x_1 x_0^2 + x_0 x_2 x_0 x_1 x_0^2)^{-1}(1 + x_0)^{-1}
\]
\[ \cdot (1 - x_2 x_0 x_1 x_0^2 + x_0 x_2 x_0 x_1 x_0^2)(1 + x_0) \] 
\[ = (1 + x_2 x_0 x_1 x_0^2 - x_0 x_2 x_0 x_1 x_0^2)(1 - x_0 + x_0^2) \] 
\[ \cdot (1 - x_2 x_0 x_1 x_0^2 + x_0 x_2 x_0 x_1 x_0^2 + x_0) \] 
\[ = (1 + x_2 x_0 x_1 x_0^2 - x_0 x_2 x_0 x_1 x_0^2) \] 
\[ \cdot (1 - x_2 x_0 x_1 x_0^2 + x_0 x_2 x_0 x_1 x_0^2) \] 
\[ + x_0 - x_0 + x_0 x_2 x_0 x_1 x_0^2 - x_0^2 + x_0^2) \] 
\[ = (1 + x_2 x_0 x_1 x_0^2 - x_0 x_2 x_0 x_1 x_0^2) \] 
\[ \cdot (1 - x_2 x_0 x_1 x_0^2 + 2x_0 x_2 x_0 x_1 x_0^2) \] 
\[ = (1 + x_0 x_2 x_0 x_1 x_0^2). \] 

Finally,

\[ [z, 1 + x_0, 1 + x_0, 1 + x_0] = (1 + x_0 x_2 x_0 x_1 x_0^2)^{-1}(1 + x_0)^{-1} \] 
\[ (1 + x_0 x_2 x_0 x_1 x_0^2)(1 + x_0) \] 
\[ = (1 - x_0 x_2 x_0 x_1 x_0^2)(1 - x_0 + x_0^2) \] 
\[ (1 + x_0 x_2 x_0 x_1 x_0^2 + x_0) \] 
\[ = (1 + x_0 x_2 x_0 x_1 x_0^2) \] 
\[ (1 + x_0 x_2 x_0 x_1 x_0^2 + x_0 - x_0 - x_0^2 + x_0^2) \] 
\[ = (1 - x_0 x_2 x_0 x_1 x_0^2) \] 
\[ (1 + x_0 x_2 x_0 x_1 x_0^2) \] 
\[ = 1. \]

This completes the proof. \( \square \)

1.3. Computer Work. This subsection provides a presentation for the Gupta and Levin example that is in a convenient form for computer calculations. We will call the group of the Gupta and Levin example \( H \), while the group we construct on the computer we will call \( G \). Consider the group \( G \) with the presentation

\[ \langle x, y, z \mid [y, z] = [x, y, y] = [x, z, z] = [x, y, z, x, x, x] = [x, y, z, x, z, y] = w^5 = 1, \text{for all words } w \text{ in } x, y, z \rangle. \]

The relators in \( G \) are relators that hold in \( H \) by Lemma 3.7, so \( H \) is a quotient of \( G \). The group \( G \) can be entered into GAP with the code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:="[f2,f3]",
```

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"[f1,f2,f2]",
"[f1,f3,f3]",
"[f1,f2,f3,f1,f1,f1]",
"[f1,f2,f3]*[f1,f3,f2]";
g:=Pq(f: Prime:=5, Exponent:=5, Relators:=rels);

When we create $G$ in GAP, it gives the output

#I Class 1 with 3 generators.
#I Class 2 with 5 generators.
#I Class 3 with 7 generators.
#I Class 4 with 10 generators.
#I Class 5 with 11 generators.
#I Class 6 with 12 generators.

From this output, we read that $G$ has pc-structure $\{3,5,7,10,11,12\}$. In addition, GAP says:

gap> g;
<pc group of size 244140625 with 12 generators>
gap> Factors(Order(g));
[ 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5 ]

This output tells us that $G$ is of order $5^{12}$.

Because all of the relators and laws we impose in $G$ are known to hold in $H$, we know that $H$ is a quotient of $G$. We have determined using GAP that $G$ is of order $5^{12}$, so $H$ has order less than or equal to $5^{12}$, not $5^{20}$ as shown earlier. Table 2 gives a (not necessarily unique) listing of pc generators for $G$, using simple commutators for consistency with later work. The weight 1 pc-generators are the same as the group generators. For higher weights, GAP was used to find a simple commutator that GAP reported was a pc-generator. For instance, when the commutator $[x,y]$ was entered into GAP, it returned $g.4$, so we know that $[x,y]$ can be considered the fourth pc-generator of $G$.

2. Restrictions on Groups with Condition $\mathcal{C}_4$ but not Condition $\mathcal{L}(\mathfrak{N}_3)$

We know that the Gupta and Levin example $H$ has order at most $5^{12}$. We also know that $H$ is a 4-Engel group and that it has an element $x$ with a normal closure
Table 2: PC-generators for $G$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[x,y]$</td>
<td>$[x,z]$</td>
<td></td>
</tr>
<tr>
<td>$[x,y,x]$</td>
<td>$[x,z,x]$</td>
<td></td>
</tr>
<tr>
<td>$[x,y,x,x]$</td>
<td>$[x,z,x,x]$</td>
<td>$[x,y,x,z]$</td>
</tr>
<tr>
<td>$[x,y,x,x,z]$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$x^H$ that is not nilpotent of class 3. We say that a group $K$ has property $\mathfrak{T}$ if $K$ is 4-Engel and there is some element $x$ of $K$ such that the normal closure $x^K$ is not nilpotent of class 3. Traustason has shown that a group with property $\mathfrak{T}$ must have at least one element of order 2 or of order 5 [47]. In this section, we find some limitations on groups that have property $\mathfrak{T}$. The goal of this section is to prove the following theorems:

**Theorem 3.8.** If $G$ is a 4-Engel group and is nilpotent of class at most 5, then the normal closure in $G$ of every element of $G$ is nilpotent of class at most 3.

**Theorem 3.9.** If $G$ is a 4-Engel group on two generators with exponent 5, then for every element $a$ in $G$, the normal closure $a^G$ is nilpotent of class at most 3.

### 2.1. Nilpotence Class Restrictions.
We begin by showing that a group with property $\mathfrak{T}$ must not be nilpotent of class 5 or smaller.

**Proof of Theorem 3.8.** Let $x, a, b, c, d$ be elements of $G$. We want to show that $x^G$ is nilpotent of class at most 3. It suffices to show that a commutator of weight 4 in the generators of $x^G$, in this case $[x^a, x^b, x^c, x^d]$, is trivial. First we calculate

$$[x^a, x^b]_{\gamma_4}(G) = [(x^b)(b^{-1}a), x^b]_{\gamma_4}(G)$$

$$= [(b^{-1}a)^{-1} x^b (b^{-1}a), x^b]_{\gamma_4}(G)$$

$$= [(b^{-1}a)^{-1} x^b, x^b]_{\gamma_4}(G)$$

$$= ([b^{-1}a]^{-1} x^b [x^b, x^b] b^{-1} a, x^b]_{\gamma_4}(G)$$

$$= ([b^{-1}a]^{-1} x^b [x^b, x^b] b^{-1} a, x^b]_{\gamma_4}(G)$$

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We have just shown that \([x^a, x^b] = [b^{-1}a, x^b]^{-1}y\), where \(y\) is in \(\gamma_4(G)\), so

\[
[x^a, x^b, x^c] = [[b^{-1}a, x^b, x^b]^{-1}y, x^c] \\
= [b^{-1}a, x^b, x^b]^{-1}, [b^{-1}a, x^b, x^b]^{-1}, y][y, x^c].
\] (20)

Because \([[b^{-1}a, x^b, x^b]^{-1}, x^c, y]\) and \([y, x^c]\) are both in \(\gamma_5(G)\),

\[
[x^a, x^b, x^c]_{\gamma_5(G)} = [[b^{-1}a, x^b, x^b]^{-1}, x^c]_{\gamma_5(G)} \\
= ([b^{-1}a, x^b, x^b]^{-1})_{\gamma_5(G)} \gamma_5(G) \\
= (b^{-1}a, x^b, x^b, (x^b)^{-1}c)_{\gamma_5(G)} \\
= [b^{-1}a, x^b, x^b, [b^{-1}c, (x^b)^{-1}]]_{\gamma_5(G)} \\
= [b^{-1}a, x^b, x^b, [b^{-1}a, x^b, x^b]^{-1}]_{\gamma_5(G)}.
\] (22)

Similarly,

\[
[x^a, x^b, x^c, x^d]_{\gamma_6(G)} = [[b^{-1}a, x^b, x^b]^{-1}, x^d]_{\gamma_6(G)} \\
= ([b^{-1}a, x^b, x^b]^{-1})_{\gamma_6(G)} \gamma_6(G) \\
= (b^{-1}a, x^b, x^b, x^d)^{-1}_{\gamma_6(G)} \\
= [b^{-1}a, x^b, x^b, (x^b)^{-1}]_{\gamma_6(G)} \\
= [b^{-1}a, x^b, x^b, [b^{-1}d, (x^b)^{-1}]]_{\gamma_6(G)}.
\] (23)
\[
[b^{-1}a, x^b, x^b, x^b, x^b][b^{-1}a, x^b, x^b, x^b, [b^{-1}d, (x^b)^{-1}]]^x^b)^{-1} \gamma_6(G)
\]

Since \(G\) is 4-Engel, we conclude that \([x^a, x^b, x^c, x^d]\) is in \(\gamma_6(G)\). Since \(G\) is nilpotent of class at most 5, we conclude that \(x^G\) is nilpotent of class at most 3. \(\square\)

Hence, an \(T\) group, if nilpotent, must be nilpotent of class at least 6.

2.2. Two Generators. In this subsection we prove Theorem 3.9.

Proof of Theorem 3.9. Let \(G\) be a 4-Engel group on generators \(a\) and \(b\) with exponent 5. Let \(\hat{G}\) be the group given by the presentation

\[
\langle a, b | w^5 = [w, u, u, u, u] = 1 \text{ for all } w \text{ and } u \text{ words in } a \text{ and } b \rangle.
\]

Note that \(G\) is a quotient of \(\hat{G}\). We will show that \(\hat{G}\) has nilpotence class 6.

If we give GAP the code:

\[
f:=\text{FreeGroup}(2);
g:=\text{Pq}(f: \text{Prime}=5, \text{Exponent}=5, \text{Identities}=[\text{function}(a,b)\text{ return LeftNormedComm([a,b,b,b,b]);end}]);
\]

then GAP returns the answer:

\#I Class 1 with 2 generators.
\#I Class 2 with 3 generators.
\#I Class 3 with 5 generators.
\#I Class 4 with 8 generators.
\#I Class 5 with 10 generators.
\#I Class 6 with 11 generators.
\#I Class 6 with 11 generators.

This output shows that \(\hat{G}\) is nilpotent of class 6, so \(G\) is also nilpotent of class 6.

Now we consider the commutator \([x, x^c, x^d, x^e]\), where \(x, c, d, \text{ and } e\) are in \(G\). Then

\[
[x, x^c, x^d, x^e] = [x, x[x, c], x^d, x^e]
\]

\[
= [[[x, [x, c]][x, x]^x^c, x^d, x^e]
\]

\[
= [[[x, [x, c]], x^d, x^e]
\]

(24)
Because $G$ is nilpotent of class at most 6 and $[[c, x], [c, x]^{-1}]$ is in $\gamma_5(G)$,

$$
[[x, [c, x]^{-1}], x^d, x^e] = [[c, x, [c, x]^{-1}], x^d, x^e] = [[c, x, x][[c, x, x], [c, x]^{-1}], x^d, x^e].
$$

Weight 4 and larger commutators in $G$ commute with each other because $G$ is nilpotent of class at most 6. Thus,

$$
[x, x^e, x^d, x^e] = [c, x, x, x^d, x^e] = [c, x, x, d^{-1}(xd), x^e] = [[c, x, x, xd][c, x, x, d^{-1}][c, x, x, d^{-1}, xd], x^e] = [[c, x, d, xd][c, x, x, x][c, x, x, d][c, x, x, d^{-1}][c, x, x, d^{-1}, d][c, x, x, d^{-1}, x][c, x, x, d^{-1}, x, d], x^e].
$$

(25)

Now we note that $[c, x, x, d][c, x, x, d^{-1}][c, x, x, d^{-1}, d] = [c, x, x, d^{-1}d] = 1$, so

$$
[x, x^e, x^d, x^e] = [[c, x, x, x][c, x, x, d][c, x, x, d^{-1}, x][c, x, x, d^{-1}, x, d], x^e].
$$

(26)

Because $G$ is nilpotent of class 6 and $[c, x, x, d^{-1}, x, d]$ is in $\gamma_6(G)$, we see that

$$
[x, x^e, x^d, x^e] = [[c, x, x, x][c, x, x, x, d][c, x, x, x, d^{-1}, x, d], x^e] = [c, x, x, x, x^e][c, x, x, d][c, x, x, d^{-1}, d][c, x, x, d^{-1}, d][c, x, x, d^{-1}, x][c, x, x, d^{-1}, x, d, x^e].
$$

(27)
Since $G$ is nilpotent of class at most 6,

$$[x, x^c, x^d, x^e] = [c, x, x, x^c][c, x, x^c, x^d][c, x, x^d, x^e]. \quad (28)$$

Consider the first factor:

$$[c, x, x, x^c] = [c, x, x, x^c][c, x, x, e^{-1}][c, x, x^c, x][c, x, x^c, x^d][c, x, x^c, x^d, x^e] \quad (29)$$

Because $G$ is 4-Engel, the second and third factors of (29) are trivial, and the seventh factor is trivial because $G$ is nilpotent of class at most 6. Thus,

$$[c, x, x, x] = [c, x, x, e^{-1}][c, x, x, e^{-1}, x] \quad (30)$$

Combining (30) with (28),

$$[x, x^c, x^d, x^e] = [c, x, x, e^{-1}, x][c, x, x, d, x^e][c, x, x, d^{-1}, x, x^e]. \quad (31)$$

Now, because $G$ is nilpotent of class 6, we use Lemma 1.2 to see that

$$[x, x^c, x^d, x^e] = [c, x, x, e^{-1}, x][c, x, x, d, x][c, x, x, d^{-1}, x, x]. \quad (32)$$

The element $x$ can be written as $a^i b^j z$, where $z$ is in $\gamma_2(G)$ and $0 \leq i, j \leq 4$. Then

$$[x, x^c, x^d, x^e] = [c, x, x, e^{-1}, x][c, x, x, d, x][c, x, x, d^{-1}, x, x]$$

$$= [c, a^i b^j z, a^i b^j z, a^i b^j z, e^{-1}, a^i b^j z][c, a^i b^j z, a^i b^j z, a^i b^j z, d, a^i b^j z] \quad (33)$$
Now,

\[
[c, x]_{\gamma_3}(G) = [c, a^i b^j z]_{\gamma_3}(G) \\
= [c, z][c, a^i b^j][c, a^i b^j, z]_{\gamma_3}(G) \\
= [c, z][c, a^i b^j]_{\gamma_3}(G).
\]  

(34)

Because \(z\) is in \(\gamma_2(G)\), the commutator \([c, z]\) is in \(\gamma_3(G)\), so

\[
[c, x]_{\gamma_3}(G) = [c, a^i b^j]_{\gamma_3}(G).
\]  

(35)

Similarly,

\[
[c, x, x]_{\gamma_4}(G) = [c, a^i b^j, a^i b^j]_{\gamma_4}(G).
\]  

(36)

Moreover,

\[
[c, x, x, x]_{\gamma_5}(G) = [c, a^i b^j, a^i b^j, a^i b^j]_{\gamma_5}(G)
\]  

(37)

and

\[
[c, x, x, d^{-1}]_{\gamma_5}(G) = [c, a^i b^j, a^i b^j, d^{-1}]_{\gamma_5}(G).
\]  

(38)

Further,

\[
[c, x, x, e^{-1}]_{\gamma_6}(G) = [c, a^i b^j, a^i b^j, a^i b^j, e^{-1}]_{\gamma_6}(G),
\]  

(39)

\[
[c, x, x, d]_{\gamma_6}(G) = [c, a^i b^j, a^i b^j, a^i b^j, d]_{\gamma_6}(G),
\]  

(40)

and

\[
[c, x, x, d^{-1}, x]_{\gamma_6}(G) = [c, a^i b^j, a^i b^j, d^{-1}, a^i b^j]_{\gamma_6}(G).
\]  

(41)

Finally,

\[
[c, x, x, e^{-1}, x]_{\gamma_7}(G) = [c, a^i b^j, a^i b^j, a^i b^j, e^{-1}, a^i b^j]_{\gamma_7}(G),
\]  

(42)

\[
[c, x, x, d, x]_{\gamma_7}(G) = [c, a^i b^j, a^i b^j, a^i b^j, d, a^i b^j]_{\gamma_7}(G),
\]  

(43)

and

\[
[c, x, x, d^{-1}, x]_{\gamma_7}(G) = [c, a^i b^j, a^i b^j, d^{-1}, a^i b^j, a^i b^j]_{\gamma_7}(G).
\]  

(44)
Because $G$ is nilpotent of class at most 6, we see $\gamma_7(G)$ is trivial, so

\[
[x, x^d, x^e] = [c, a^ib^j, a^ib^j, a^ib^j, e^{-1}, a^ib^j][c, a^ib^j, a^ib^j, a^ib^j, d, a^ib^j]
\cdot [c, a^ib^j, a^ib^j, d^{-1}, a^ib^j, a^ib^j].
\] (45)

Now (32) holds for all $x$ in $G$, including $a^ib^j$, so

\[
[x, x^d, x^e] = [a^ib^j,(a^ib^j)^c,(a^ib^j)^d,(a^ib^j)^e].
\] (46)

We conclude that we may assume that $z$ is trivial.

Now we can use GAP to show that the normal closure of every element in $\hat{G}$ of the form $a^ib^j$, where $0 \leq i, j \leq 4$, is nilpotent of class at most 3. This result is seen by the following output from GAP:

```
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^3]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^4]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1*g.2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^2*g.2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^3*g.2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^4*g.2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.2^2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1*g.2^2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^2*g.2^2]))); 3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^3*g.2^2]))); 3
```

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Thus, the normal closure of every element of $\hat{G}$ is nilpotent of class at most 3. Because $G$ is a quotient of $\hat{G}$, the normal closure of every element of $G$ is also nilpotent of class at most 3. This completes the proof of Theorem 3.9. □

3. Nickel Counterexample

The example by Gupta and Levin suffices to show that condition $\mathcal{E}_4$ is not equivalent to condition $\mathcal{E}(\mathfrak{N}_3)$, but it only provides a counterexample when an element of order 5 is present. In [5], Nickel demonstrates how to input in a computer the largest nilpotent quotient of $E(i, n)$, the largest $n$-Engel group on $i$ generators. One of the groups he uses as an example is $E(3, 4)$, for which he gives
the nilpotence class, Hirsch length, and structure of the lower central series. In a comment in [62], Traustason says that Nickel’s paper provides an example of a 4-Engel 2-group that contains an element whose normal closure in the group is not nilpotent of class 3. Using GAP and ANUPQ, we compute the largest quotient of $E(3,4)$ that is nilpotent of class 6 and is a 2-group. The nilpotence class bound is set to 6 because a 4-Engel group that is nilpotent of class 5 will satisfy condition $L(N_3)$, so we need a class bound of at least 6, and we set it as small as possible to save computer time and produce a smaller group. The following code generates the example that Traustason refers to and computes the nilpotence class of the normal closure of the first generator of the resulting group:

```plaintext
RequirePackage("anupq");
f:=FreeGroup(3);
g:=Pq(f: Prime:=2, ClassBound:=6, Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b]); end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

The output from generating this group $g$ in GAP and then asking GAP for more information is the following:

```
#I Class 1 with 3 generators.
#I Class 2 with 9 generators.
#I Class 3 with 23 generators.
#I Class 4 with 55 generators.
#I Class 5 with 121 generators.
#I Class 6 with 212 generators.
gap> c;
4
```

As the computer output shows, the normal closure of the first generator of group $g$ has nilpotence class 4 rather than 3, so this group $g$ is in $E_4$ but not in $L(N_3)$. Note that $g$ has order $2^{212}$. 

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4. Smaller 2-group

In this section we construct a much smaller 4-Engel 2-group that does not satisfy Condition $\mathcal{L}(\mathfrak{N}_3)$ than the example mentioned by Traustason.

**Theorem 3.10.** There is a 4-Engel 2-group of order $2^{26}$ that does not satisfy condition $\mathcal{L}(\mathfrak{N}_3)$.

This smaller 2-group example is the group with the presentation $G = \langle x, y, z \mid x^4 = y^2 = z^2 = [y, z, y] = [x, y, z, z] = [y, z, x, x] = [x, z, x, y] = \gamma_7((x, y, z)) = w^8 = [u, w] = 1, \text{ for all words } u \text{ and } w \text{ in } x, y, z \rangle$. It is created in GAP by the computer code:

```gap
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f1^4",
"f2^2",
"f3^2",
"[f2,f3,f2]",
"[f1,f2,f3,f3]",
"[f2,f3,f1,f1,f1]",
"[f1,f2,f3,f1,f2]",
];
G:=Pq(f: Prime:=2, Exponent:=8, ClassBound:=6, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(G,Subgroup(G,[G.1])));
```

The $p$-class bound is set to 6 because a larger bound would result in a larger group, while a smaller $p$-class bound would mean the example would satisfy condition $\mathcal{L}(\mathfrak{N}_3)$. GAP’s output from this code is the following:

```
#I Class 1 with 3 generators.
#I Class 2 with 7 generators.
#I Class 3 with 13 generators.
#I Class 4 with 20 generators.
#I Class 5 with 25 generators.
#I Class 6 with 26 generators.
gap> G;
<pc group of size 67108864 with 26 generators>
gap> c;
4
```
This group $G$ has the pc structure $\{3, 7, 13, 20, 25, 26\}$ and order $2^{26}$, so it is a 2-group. $G$ is also nilpotent of class 6. The normal closure of the element $G.1$, which is the first generator of $G$, is nilpotent of class 4, not class 3. These observations complete the proof of Theorem 3.10.

It should be noted that $2^{26}$ is smaller than $5^{12}$, so the smallest known 4-Engel group that does not satisfy condition $L(N_3)$ is this 2-group $G$. This example was found by trial and error, as follows. We start with the free group $f$ on three generators. Choosing the exponent to be 2 or 4 gives a group in which the normal closure of $f.1$ is nilpotent of class 3, so the exponent is fixed at 8. We try introducing the possible relator $G.1^2$, and find that this relator makes the nilpotence class of the normal closure of $G.1$ be 3. The relator $G.1^4$ does not give nilpotence class 3 for the normal closure of $G.1$, so we leave it among the relators. Then we try $G.2^2$, and again the nilpotence class of the normal closure of $G.1$ is larger than 3. The possible relator $G.3^2$ gives the same result. Next a possible weight two commutator is tested. The possible weight two commutators are $[f.1, f.2], [f.1, f.3]$, and $[f.2, f.3]$. Introducing any of these weight two commutators again gives a nilpotence class of 3 for the normal closure of $G.1$, so we do not include any weight two relators in the group $G$. Then we consider all weight three commutators in the generators of $G$. Again, we try the possible relators of weight three one at a time, and keep those that do not shrink the nilpotence class of the normal closure of $G.1$ but do make the pc-structure smaller. We repeat this process for possible relators of weight 4, 5, and 6. While we have not proved that the group $G$ so constructed is the smallest example of a 4-Engel group that has an element whose normal closure is nilpotent of class exactly 4, we believe that it is close to being the smallest such group.

5. Larger $n$ and Other Primes

Similar experimentation produces other examples of $n$-Engel groups that contain an element whose normal closure is not nilpotent of class $n - 1$, that is, do
not satisfy Condition $\mathcal{L}(\mathcal{N}_{n-1})$, for larger values of $n$. In each of the groups discussed in this section, the normal closure of first generator is not nilpotent of class $n - 1$.

The general method for constructing the examples in this section is to start with a group with the lowest possible nilpotence class $c$ and least exponent that is $n$-Engel, and check if the normal closure of the first generator is nilpotent of class $c - 1$. If so, first weight two commutators are added as relators to make the second part of the pc structure as small as possible. Then weight three commutators are considered, and the process is repeated until the weight reaches the $p$-class. This procedure might not produce for each $p$ the smallest $p$-group that has condition $\mathcal{E}_n$ but not condition $\mathcal{L}(\mathcal{N}_{n-1})$ for the desired integer $n$, but it does produce reasonably small groups. The examples in this section prove the following two theorems.

**Theorem 3.11.** There exist 5-Engel and 6-Engel 5-groups that do not satisfy conditions $\mathcal{L}(\mathcal{N}_4)$ and $\mathcal{L}(\mathcal{N}_5)$ respectively. There also exist 5-Engel and 6-Engel 2-groups that do not satisfy conditions $\mathcal{L}(\mathcal{N}_4)$ and $\mathcal{L}(\mathcal{N}_5)$ respectively.

Hence condition $\mathcal{E}_5$ does not imply condition $\mathcal{L}(\mathcal{N}_4)$ and condition $\mathcal{E}_6$ does not imply condition $\mathcal{L}(\mathcal{N}_5)$.

Theorem 3.11 is new, but Vaughan-Lee [4] exhibits a 5-group that is 5-Engel but that has a normal closure that is not 4-Engel, so that normal closure cannot be nilpotent of class 4. Vaughan-Lee’s example is given in detail in Section 1 of Chapter 5.

**Theorem 3.12.** There exist 5-Engel 3-groups and 7-groups that do not satisfy condition $\mathcal{L}(\mathcal{N}_4)$.

Theorem 3.12 shows that for 3-groups and 7-groups there exists an $m$, namely $m = 5$, such that condition $\mathcal{E}_m$ does not imply condition $\mathcal{L}(\mathcal{N}_{m-1})$. We show how to construct examples that justify Theorems 3.11 and 3.12 in the next two subsections.

**5.1. 5-Engel groups.** A reasonably small $\mathcal{E}_5$ 5-group $G_5$ without $\mathcal{L}(\mathcal{N}_4)$ is given by the presentation $G_5 = \langle x, y, z \mid [y, z] = [x, y, y] = [x, y, z] = [x, z, z] = \cdots \rangle$. 

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\[ \gamma_8(\langle x, y, z \rangle) = w^5 = 1 \text{ for all words } w \text{ in } x, y, z. \] This group is nilpotent of class 7, has the pc structure \{3, 5, 7, 10, 11, 12, 13\}, and has order \(5^{13}\). The GAP code for this group is

```gap
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
    "[f2,f3]",
    "[f1,f2,f2]",
    "[f1,f2,f3]",
    "[f1,f3,f3]"];
g:=Pq(f: Prime:=5, Exponent:=5, ClassBound:=7, Relators:=rels);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

GAP's output for \(G_5\) is the following:

```
gap> g;
<pc group of size 1220703125 with 13 generators>
gap> c;
5
```

\(G_5\) is 5-Engel because the quotient of \(G_5\) obtained by adding the 5-Engel law has the same order as \(G_5\).

A reasonably small \(C_5\) 2-group \(G_2\) without \(L(\mathfrak{M}_4)\) has the presentation

\[
G_2 = \langle x, y, z \mid x^4 = y^2 = z^2 = [y, z, z] = [x, y, y] = [x, z, x]^2 = [y, z, x]^2 = [x, y, x, x] = [x, y, z, z] = [x, z, z, x] = [y, z, x, x] = [x, y, x, x, x] = \gamma_8(\langle x, y, z \rangle) = [u, w] = w^8 = 1, \text{ for all } u \text{ and } w \text{ words in } x, y, z. \]

The group \(G_2\) is input in GAP with the code:

```gap
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
    "f1^4",
    "f2^2",
    "f3^2",
    "[f2,f3,f3]",
    "[f1,f2,f2]",
    "[f1,f2,f3]",
    "[f1,f3,f1]^2",
    "[f2,f3,f1]^2",
    "[f1,f3,f1]^2",
    "[f1,f2,f1,f1]",
];
g:=Pq(f: Prime:=5, Exponent:=5, ClassBound:=7, Relators:=rels);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

GAP's output for \(G_2\) is:

```
gap> g;
<pc group of size 2048 with 7 generators>
gap> c;
3
```

\(G_2\) is 3-Engel because the quotient of \(G_2\) obtained by adding the 3-Engel law has the same order as \(G_2\).
g:=Pq(f: Prime:=2, Exponent:=8, ClassBound:=7, Relators:=rels, Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]); end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));

Here the $p$-class bound is set to 7 because in a group that is nilpotent of class 6 the normal closure of every element in the group is nilpotent of class 5, and the nilpotence and $p$-class bounds usually match up rather well in the groups we examine. The exponent is set to 8 because a smaller exponent does not give the desired nilpotence class for a normal closure. GAP gives the following output for this code:

```gap
#I Class 1 with 3 generators.
#I Class 2 with 7 generators.
#I Class 3 with 12 generators.
#I Class 4 with 17 generators.
#I Class 5 with 21 generators.
#I Class 6 with 23 generators.
#I Class 7 with 24 generators.
gap> g;
<pc group of size 16777216 with 24 generators>
gap> Order(g);
16777216
gap> NilpotencyClassOfGroup(g);
7
gap> c;
5
```

Note that $G_2$ has order $2^{24}$.

5.2. 6-Engel Groups. The group given to GAP by the code:
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f1^5",
"f2^5",
"f3^5",
"[f2,f3]",
"[f1,f2,f2]",
"[f1,f3,f3]",
"[f1,f2,f1,f2]",
"[f1,f3,f1,f3]",
"[f1,f3,f2,f2]",
"[f1,f2,f1,f1,f1]",
"[f1,f3,f1,f1,f1]"];
g:=Pq(f: Prime:=5, Exponent:=25, ClassBound:=8, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));

is the 6-Engel group \( H_5 \) given by

\[ \langle x, y, z \mid x^5 = y^5 = z^5 = [y,z] = [x,y,y] = [x,z,z] = [x,y,x,y] = 
\]
\[ [x,z,x,z] = [x,z,y,y] = [x,y,x,x,x] = [x,y,x,z,z] = [x,z,x,x,x] = \gamma_8(\langle x,y,z \rangle) = 
\]
\[ [w,u] = w u^{25} = 1 \text{ for all words } w, u \text{ in } x, y, z \].

Then \( H_5 \) is an \( E_6 \) 5-group of order \( 5^{15} \), but not a \( L(N_5) \) group, as is seen by the output:

\[
\text{#I Class 1 with 3 generators.}
\text{#I Class 2 with 5 generators.}
\text{#I Class 3 with 7 generators.}
\text{#I Class 4 with 10 generators.}
\text{#I Class 5 with 12 generators.}
\text{#I Class 6 with 13 generators.}
\text{#I Class 7 with 14 generators.}
\text{#I Class 8 with 15 generators.}
\text{gap> g;}
\text{<pc group of size 30517578125 with 15 generators>}
\text{gap> c;}
6
\]

This group \( H_5 \) is not of exponent 5, because if we have GAP compute the exponent 5 quotient of \( H_5 \), the resulting group has smaller order.

Similarly, the 6-Engel 2-group \( H_2 \) given by the GAP code
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f1^4",
"f2^2",
"f3^2",
"[f2,f3]",
"[f1,f2,f3]",
"[f1,f2,f1,f2]",
"[f1,f2,f2,f2]",
"[f1,f3,f1,f3]",
"[f1,f3,f3,f2]",
"[f1,f3,f3,f3]",
"[f1,f2,f2,f1,f3]",
"[f1,f3,f2,f1,f1,f1]",
];
g:=Pq(f: Prime:=2, Exponent:=8, ClassBound:=8, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));

has presentation \langle x,y,z \mid x^4 = y^2 = z^2 = [y,z] = [x,y,z] = [x,y,x,y] = [x,y,y,y] = [x,z,x,z] = [x,z,z,y] = [x,y,y,x,z] = [x,z,y,x,x,x] = \gamma_8((x,y,z)) = w^8 = [u,6w] = 1 \text{ for all words } u \text{ and } w \text{ in } x,y,z \rangle.

GAP gives the output:

#I Class 1 with 3 generators.
#I Class 2 with 6 generators.
#I Class 3 with 10 generators.
#I Class 4 with 13 generators.
#I Class 5 with 17 generators.
#I Class 6 with 21 generators.
#I Class 7 with 22 generators.
#I Class 8 with 23 generators.
gap> g;
<pc group of size 8388608 with 23 generators>
gap> Order(g);
8388608
gap> NilpotencyClassOfGroup(g);
8
gap> c;
6
This group $H_2$ has order $2^{23}$, and does not satisfy $\mathfrak{L}(\mathfrak{N}_5)$. The examples just discussed in subsections 5.1 and 5.2 justify Theorem 3.11.

5.3. Other Primes. As noted earlier, it is known that a group must have an element of order 2 or 5 to be 4-Engel but contain an element whose normal closure is not nilpotent of class 3, but it is shown here that the same restriction does not hold for 5-Engel groups that contain an element whose normal closure is not nilpotent of class 4. A reasonably small 5-Engel 3-group $G_3$ of order $3^{12}$ that contains an element whose normal closure in not nilpotent of class 4 is given by GAP code:

```gap
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
  "f2^3",
  "f3^3",
  "[f2,f3]",
  "[f1,f2,f1]",
  "[f1,f2,f2]",
  "[f1,f2,f3]",
  "[f1,f3,f3]"];
g:=Pq(f: Prime:=3, Exponent:=9, ClassBound:=7, Relators:=rels,
  Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
  end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
and the presentation $\langle x, y, z \mid y^3 = z^3 = [y, z] = [x, y, x] = [x, y, y] = [x, y, z] = [x, z, z] = \gamma_8(\langle x, y, z \rangle) = w^9 = [u, w^5] = 1$ for all words $u$ and $w$ in $x, y, z$).

GAP gives as output:

#I Class 1 with 3 generators.
#I Class 2 with 6 generators.
#I Class 3 with 7 generators.
#I Class 4 with 8 generators.
#I Class 5 with 10 generators.
#I Class 6 with 11 generators.
#I Class 7 with 12 generators.
gap> g;
<pc group of size 531441 with 12 generators>
gap> Order(g);
531441
gap> NilpotencyClassOfGroup(g);
```
This 5-Engel group $G_3$ of order $3^{12}$ does not satisfy $L(E_4)$. A $\mathfrak{E}_5$ 7-group $G_7$ of order $7^{16}$ can be obtained by the GAP code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
    "[f2,f3]",
    "[f1,f2,f2]",
    "[f1,f2,f3]",
    "[f1,f3,f3]",
    "[f1,f2,f1,f2]",
    "[f1,f3,f1,f3]"];
g:=Pq(f: Prime:=7, Exponent:=7, ClassBound:=7, Relators:=rels,
    Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
    end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

and is given by the presentation

$$\langle x, y, z, \mid [y, z] = [x, y, y] = [x, z, z] = [x, y, x, y] = [x, z, x, z] = \gamma_8(\langle x, y, z \rangle) = w^7 = [u, 5w] = 1 \text{ for all words } u \text{ and } w \text{ in } x, y, z \rangle.$$

When this code is input into GAP, the results are:

```
#I Class 1 with 3 generators.
#I Class 2 with 5 generators.
#I Class 3 with 7 generators.
#I Class 4 with 10 generators.
#I Class 5 with 14 generators.
#I Class 6 with 15 generators.
#I Class 7 with 16 generators.
g; 
<pc group of size 33232930569601 with 16 generators>
Order(g); 
33232930569601
NilpotencyClassOfGroup(g); 
7
c; 
5
```

This 5-Engel group $G_7$ of order $7^{16}$ does not satisfy $L(\mathfrak{E}_4)$. 

---
The examples in subsection 5.3 justify Theorem 3.12.
Chapter 4

Examples Where $\mathcal{E}_n$ Does Not Imply $\mathcal{L}(\mathfrak{N})$

The examples in Chapter 3 show that a group can satisfy condition $\mathcal{E}_n$, the condition of being an $n$-Engel group, but not satisfy condition $\mathcal{L}(\mathfrak{N}_{n-1})$, meaning they contain an element whose normal closure is not nilpotent of class 3. However, the examples in Chapter 3 all satisfy $\mathcal{L}(\mathfrak{N})$, the condition of every normal closure being nilpotent, because the groups are nilpotent. Gupta and Levin in [3] give a family of examples that show that there exist groups that are $n$-Engel but in which there is an element with a non-nilpotent normal closure in the group. We follow their construction.

**Construction 1 (Construction of $M_p$).** Let $p$ be an odd prime. Let $G_p$ be the relatively free group on countably many generators $\{x_0, x_1, \ldots\}$ with exponent $p$ and of nilpotence class 2. Let $M_p$ be the set of $2 \times 2$ matrices over the group ring $\mathbb{Z}_p G_p$ of the form $(g^{0 \ 0 \ r \ 1})$, where $g$ is in $G_p$ and $r$ is in $\mathbb{Z}_p G_p$.

**Lemma 4.1.** (Theorem 1 of Gupta and Levin [3]) $M_p$ has the following properties:

1. $M_p$ is a group.
2. $M_p$ has exponent $p^2$.
3. $\gamma_i(M_p)$ is contained in $\left\{ \left( \begin{array}{cc} \gamma_i(G_p) & 0 \\ \mathbb{Z}_p G_p & 1 \end{array} \right) \right\}$ for each positive integer $i$.
4. $M_p$ is $(p + 2)$-Engel.
5. $M_p$ is not in $\mathcal{L}(\mathfrak{N})$.

**Proof.** (1) Because the set of all invertible $2 \times 2$ matrices over $\mathbb{Z}_p G_p$ is a group, we need to show that the set $M_p$ is closed under multiplication and inversion.
To show that $M_p$ is closed under multiplication, let $g_1$ and $g_2$ be elements of $G$ and let $r_1$ and $r_2$ be elements of $\mathbb{Z}_pG$. Then
\[
\begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix} = \begin{pmatrix} g_1g_2 & 0 \\ r_1g_2 + r_2 & 1 \end{pmatrix} \in M_p.
\] (47)

Also, 
\[
\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix} \cdot \begin{pmatrix} g^{-1} & 0 \\ -rg^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\] (48)

and
\[
\begin{pmatrix} g^{-1} & 0 \\ -rg^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\] (49)

so
\[
\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}^{-1} = \begin{pmatrix} g^{-1} & 0 \\ -rg^{-1} & 1 \end{pmatrix} \in M_p.
\] (50)

Thus $M_p$ is a group.

(2) To show that the exponent of $M_p$ is $p^2$, we consider the subset $N_p$ of $M_p$ defined to be the matrices of the form \((\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix})\), where $r \in \mathbb{Z}_pG$. We notice that
\[
\begin{pmatrix} 1 & 0 \\ r_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ r_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r_1 + r_2 & 1 \end{pmatrix},
\] (51)

so $N_p$ is closed under multiplication. Also,
\[
\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix},
\] (52)

so $N_p$ is closed under inversion and thus is a subgroup of $M_p$. We also observe that $N_p$ is abelian because $r_1 + r_2 = r_2 + r_1$ for all $r_1, r_2 \in \mathbb{Z}_pG$.

From (47) we see that
\[
\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}^p = \begin{pmatrix} 1 & 0 \\ pr & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\] (53)
so \( N_p \) is of exponent \( p \). Again using \((47)\) we see that
\[
\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}^p = \begin{pmatrix} g^p & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \hat{r} & 1 \end{pmatrix},
\]
where \( \hat{r} = r \sum_{i=0}^{p-1} g^i \) is in \( \mathbb{Z}_p \). Therefore the \( p \)th power of any element of \( M_p \) is in \( N_p \), and hence \( M_p \) is of exponent \( p^2 \).

(3) Because the remaining properties are commutator properties, we first calculate a general commutator in \( M_p \). If \( g_1 \) and \( g_2 \) are elements of \( G_p \) and \( r_1 \) and \( r_2 \) are elements of \( \mathbb{Z}_p \), then
\[
\left[ \begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix}, \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix} \right] = \begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix} \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix}
= \begin{pmatrix} g_{i-1}^{-1} & 0 \\ -r_1g_{i-1}^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_2^{-1} & 0 \\ -r_2g_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix} \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix}
= \begin{pmatrix} g_{i-1}^{-1}g_1^{-1} & 0 \\ -r_1g_{i-1}^{-1}g_2^{-1} - r_2g_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_1g_2 & 0 \\ r_1g_2 + r_2 & 1 \end{pmatrix}
= \begin{pmatrix} g_{i-1}^{-1}g_1^{-1}g_2^{-1} & 0 \\ -r_1[g_1, g_2] - r_2g_1g_2^{-1} + r_1g_2 + r_2 & 1 \end{pmatrix}.
\]

By induction, we see that \( \gamma_i(M_p) \) is contained in \( \left( \frac{\gamma_i(G_p) \mathbb{Z}_p}{\mathbb{Z}_p G_p} \right) \).

(4) Let \( A \) and \( B \) be elements of \( M_p \). The commutator \( [A, 2B] \) is in \( \gamma_3(M_p) \).

From (3), it follows that \( [A, 2B] \) is in \( \left( \frac{\gamma_3(G_p) \mathbb{Z}_p}{\mathbb{Z}_p G_p} \right) \). However, \( G_p \) is nilpotent of class 2, so \( [A, 2B] \) is in \( N_p \), and hence has the form \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) for \( s \) in \( \mathbb{Z}_p \). The matrix \( B \) has the form \( \begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix} \), where \( g \) is in \( G_p \) and \( r \) is in \( \mathbb{Z}_p \). We prove by induction that \( [A, iB] = \begin{pmatrix} 1 & 0 \\ s(g-1)^{-i} & 1 \end{pmatrix} \) for \( i \geq 2 \). We already know that \( [A, 2B] = \begin{pmatrix} 1 & 0 \\ s(g-1)^{1} & 1 \end{pmatrix} \). We now assume that \( [A, iB] = \begin{pmatrix} 1 & 0 \\ s(g-1)^{i-2} & 1 \end{pmatrix} \) and show that \( [A, i+1B] = \begin{pmatrix} 1 & 0 \\ s(g-1)^{i-1} & 1 \end{pmatrix} \). We see that
\[
[A, i+1B] = [A, iB, B]
= \left[ \begin{pmatrix} 1 & 0 \\ s(g-1)^{i-2} & 1 \end{pmatrix}, \begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix} \right]
= \begin{pmatrix} 1 & 0 \\ s(g-1)^{i-1} & 1 \end{pmatrix}.
\]
\[
\begin{align*}
&= \left(-s(g-1)^{-2} \begin{bmatrix} 1, g \\ 0 \end{bmatrix} - r 1^g + s(g-1)^{-2}g + r 1 \right) \\
&= \left(-s(g-1)^{-2} - r + s(g-1)^{-2}g + r 1 \right) \\
&= \left(-s(g-1)^{-2} + s(g-1)^{-2}g 1 \right) \\
&= \left(s(g-1)^{-1}g 1 \right).
\end{align*}
\]

This calculation completes the induction. Therefore

\[
[A, p+2B] = \left( \frac{1}{s(g-1)^p} 0 \right).
\]

Since \((g-1)^p = g^p - 1\) in \(\mathbb{Z}_pG_p\) and \(G\) has exponent \(p\), we see that

\((g-1)^p = 0\), so

\[
[A, p+2B] = \left( \frac{1}{0^0} \right),
\]

and we conclude that \(M_p\) is \((p + 2)\)-Engel.

(5) For each \(x_i\) a generator of \(G_p\), we define \(X_i \) in \(M_p\) to be \(\left( \begin{smallmatrix} x_0 & 0 \\ 1 & 1 \end{smallmatrix} \right)\), and then let

\(Y_i = [X_0, X_i]\). Also, let \(Y\) be \(\left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)\). We now show by induction on \(i\) that

\([Y, Y_1, \ldots, Y_i] = \left( \begin{smallmatrix} 1 & 0 \\ u_n & 1 \end{smallmatrix} \right)\), where \(u_n = ([x_0, x_1] - 1) \cdots ([x_0, x_n] - 1)\).

We begin by calculating \(Y_i\). Using (55),

\[
Y_i = [X_0, X_i] \\
= \left[ \begin{bmatrix} x_0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} x_i & 0 \\ 1 & 1 \end{bmatrix} \right] \\
= \left( -[x_0, x_i] \begin{bmatrix} x_0 & x_i \\ 1 & 1 \end{bmatrix} \right) \\
= \left( \begin{bmatrix} x_0, x_i \\ y_i \end{bmatrix} 0 \right)
\]

where \(y_i = -[x_0, x_i] - x_0^{x_i} + x_i + 1\).

Now
\[ [Y, Y_i] = \left[ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} [x_0, x_1] & 0 \\ y_i & 1 \end{pmatrix} \right] \]
\[ = \left( -[1, [x_0, x_1]] - y_i 1^{[x_0, x_1]} + [x_0, x_1] + y_i 1 \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
\[ = \left( -1 - y_i + [x_0, x_1] + y_i 1 \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 \\ u_i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] \quad (60)

We now assume that \([Y, Y_1, \ldots, Y_i] = \left( \begin{pmatrix} 1 \\ u_i \end{pmatrix} \right)\) and show that
\[ [Y, Y_1, \ldots, Y_{i+1}] = \left( \begin{pmatrix} 1 \\ u_{i+1} \end{pmatrix} \right) \]. Using (55) once again,

\[ [Y, Y_1, \ldots, Y_{i+1}] = [[Y, Y_1, \ldots, Y_i], Y_{i+1}] \]
\[ = \left[ \begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}, \begin{pmatrix} [x_0, x_{i+1}] & 0 \\ y_{i+1} & 1 \end{pmatrix} \right] \]
\[ = \left( -u_i [1, [x_0, x_{i+1}]] - y_{i+1} 1^{[x_0, x_{i+1}]} + u_i [x_0, x_{i+1}] + y_{i+1} 1 \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
\[ = \left( -u_i - y_{i+1} + u_i [x_0, x_{i+1}] + y_{i+1} 1 \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] \quad (61)
\[ = \begin{pmatrix} u_i(-1 + [x_0, x_{i+1}]) \\ u_{i+1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

This calculation completes the induction. We now consider
\[ u_n = ([x_0, x_1] - 1) \cdots ([x_0, x_n] - 1). \] We can expand this product to see that
\[ u_n = (-1)^n + (-1)^{n-1} S_1 + (-1)^{n-2} S_2 + (-1)^{n-3} S_3 + \cdots - S_{n-1} + S_n, \]
where
\[ S_1 = \sum_{i=1}^{i=n} [x_0, x_i], \]
Because $G_p$ is defined as the relatively free group on countably many generators $\{x_0, x_1, \ldots\}$ with exponent $p$ and nilpotence class 2, the $S_j$ are disjoint sums and $S_1$ does not equal 0. We conclude that $u_n$ is not zero.

Now, we note that $[Y, Y_1]$ and $Y_k$ for every positive integer $k$ are in $\gamma_1(X_0^{M_p}) = X_0^{M_p}$. If $[Y, Y_1, \ldots, Y_{k-1}] \in \gamma_{k-1}(X_0^{M_p})$, then $[Y, Y_1, \ldots, Y_k] \in \gamma_k(X_0^{M_p})$. We conclude by induction that $[Y, Y_1, \ldots, Y_i] \in \gamma_i(X_0^{M_p})$ for every positive integer $i$, so $\gamma_i(X_0^{M_p})$ contains a non-trivial element. Thus no term $\gamma_i(X_0^{M_p})$ of the lower central series of $X_0^{M_p}$ is trivial, so $X_0^{M_p}$ is not nilpotent. We conclude that $M_p$ contains an element, $X_0$, whose normal closure in $M_p$ is not nilpotent, and therefore $M_p$ does not satisfy condition $\mathfrak{L}(\mathfrak{N})$.

\[\text{□} \]

**Theorem 4.2.** (Gupta and Levin [3]) For every odd prime $p$, there exists a $p$-group that is $(p + 2)$-Engel, but that has an element whose normal closure is not nilpotent.

**Proof.** This follows immediately from Lemma 4.1. \[\text{□} \]

Note that $M_3$ is a 5-Engel group that does not satisfy condition $\mathfrak{L}(\mathfrak{N})$. However, all 4-Engel groups satisfy condition $\mathfrak{L}(\mathfrak{N}_4)$ [4]. Note also that the groups
$M_p$ are not finitely generated. It remains an open question if every finitely generated $n$-Engel group for $n$ greater than 4 satisfies condition $\mathcal{L}(\mathfrak{N})$. 
Examples In Which \(E_n\) Does Not Imply \(L(E_{n-1})\)

In Chapter 3, we considered groups that show that condition \(E_n\) and condition \(L(N_{n-1})\) are not equivalent. In this chapter, we investigate groups that show that \(E_n\) and \(L(E_{n-1})\) are not equivalent either. We start by discussing an example by Vaughan-Lee [4] of a group that is 5-Engel but in which there is an element that has a normal closure that is not 4-Engel. Then we consider an example published by Rips and Shalev in [6]. For every odd prime \(p\), they construct a \(p\)-group that demonstrates that there exists a natural number \(n\) such that an \(n\)-Engel \(p\)-group does not necessarily satisfy condition \(L(E_{n-1})\). We follow a hint in their paper to construct a 2-group that shows that there exists a natural number \(n\) such that a \(n\)-Engel 2-group does not necessarily satisfy condition \(L(E_{n-1})\). For each prime \(p\), we then calculate an upper bound on the minimal value of \(n\) for which such examples exist.

1. Vaughan-Lee’s Example

Vaughan-Lee notes in [4] that the largest possible nilpotent 5-Engel group generated by two elements of order 3 contains an element whose normal closure is not 4-Engel. We construct the largest 3-group quotient of Vaughan-Lee’s group using GAP and ANUPQ by the following code:

```gap
RequirePackage("anupq");
f:=FreeGroup(2);
rels:=["f1^3", "f2^3"];
g:=Pq(f: Prime:=3, Relators:=rels,
   Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
   end]);
```

GAP then gives the following information:
As the GAP output shows, the commutator \([a, [b, a], [b, a], [b, a]]\) is not trivial. Because \([b, a]\) is in the normal closure of \(a\), the normal closure of \(a\) is not 4-Engel.

The group \(g\) also can be given by the presentation
\[
\langle x, y \mid x^3 = y^3 = [u, w] = 1, \text{ where } u \text{ and } w \text{ are any words in } x \text{ and } y \rangle.
\]
The order of \(g\) is \(3^{17}\). It has pc structure \([2, 3, 5, 7, 11, 12, 14, 15, 17]\). Because the order is the same as that for Vaughan-Lee’s group, this group \(g\) is the same as his group.

We now consider \(h\), the quotient of \(g\) we obtain when we impose the additional relator \([y, x, x, y, x, y, x, y, y]\). This group \(h\) is input with the following GAP script:

```gap
RequirePackage("anupq");
f:=FreeGroup(2);
rels:=[
"f1^3",
"f2^3",
"[f2,f1,f1,f2,f1,f2,f1,f2,f2]"];

h:=Pq(f: Prime:=3, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
end]);

n:=LeftNormedComm([h.2,h.1]);
m:=LeftNormedComm([h.1,n,n,n,n]);
```

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The group \( h \) is 5-Engel and has an element whose normal closure is not 4-Engel. Also, \( h \) is of order \( 3^{16} \). The following is a GAP log showing these properties:

```gap
#I Class 1 with 2 generators.
#I Class 2 with 3 generators.
#I Class 3 with 5 generators.
#I Class 4 with 7 generators.
#I Class 5 with 11 generators.
#I Class 6 with 12 generators.
#I Class 7 with 14 generators.
#I Class 8 with 15 generators.
#I Class 9 with 16 generators.
gap> m;
f16^2
```

As far as we know, this group \( h \) is a new example of a 5-Engel group that contains an element whose normal closure is not 4-Engel.

## 2. Rips and Shalev Counterexample

The above example of Vaughan-Lee and the group \( h \) from the previous section show that condition \( \mathfrak{E}_5 \) does not imply condition \( \mathcal{L}(\mathfrak{E}_4) \). However, they are 3-groups, and leave open the question of how these conditions relate for groups without elements of order 3. In this section we explain the details of a family of examples by Rips and Shalev \[6\]. For each odd prime \( p \), a group from this family shows that there must exist some positive integer \( n_p \) such that an \( n_p \)-Engel \( p \)-group exists that has an element whose normal closure is not \((n_p - 1)\)-Engel. We also construct a similar 2-group. The 2-group construction follows a hint in the paper of Rips and Shalev, and our examination of the construction for odd primes provides many more details than their paper does.

### 2.1. Definitions

We need a few definitions before we can construct the examples of Rips and Shalev.
**Definition 5.1.** Let $F_t$ be a free group of rank $t$, with $t > 1$. Then a *sequence of basic commutators* in $F_t$ is an infinite sequence $c_1, c_2, \ldots$ of elements of $F_t$ such that:

1. each $c_i$ has associated with it a positive integer $w_i$ called its weight;
2. the $c_i$ are ordered by weight, meaning that if $j > i$, then $w_j \geq w_i$;
3. the basic commutators of weight 1, namely $c_1, \ldots, c_t$, are the free generators of $F_t$ arranged in some order;
4. if $w_k > 1$, then $c_k$ is described explicitly by $[c_j, c_i]$, where $j > i$ and $w_j + w_i = w_k$. Also, if $w_j > 1$, then there exist $q < j$ and $r \leq i$ such that $c_j = [c_q, c_r]$.

A **basic commutator** is any $c_i$ in some sequence of basic commutators for $F_t$ for some integer $t > 1$.

Any finitely generated free group has many different sequences of basic commutators, based on choices of the order of basic commutators of a given weight. A choice made at one weight will affect what choices are available at later weights.

We will often use basic commutators in the relatively free group of exponent $e$, nilpotence class $c$, and rank $d$. We will refer to this relatively free group as $H_{d,e,c}$. A sequence of basic commutators for $H_{d,e,c}$ is derived from a sequence of basic commutators of $F_d$. It contains images under the natural map of all the basic commutators of $F_d$ of weight at most $c$. By Theorem 11.2.4 of [63], if $\{c_1, \ldots, c_s\}$ are the elements in an ordered sequence of basic commutators for $F_d$ that are of weight $c$ or less, then an arbitrary element $f$ of $F_d$ has a unique representation relative to this sequence of basic commutators:

$$f = c_1^{e_1} \cdots c_s^{e_s} \mod \gamma_{s+1}(F_d),$$

(63)
where \( e_i \) is a non-negative integer. In \( H_{d,e,c} \), each element \( h \) has a representation (not necessarily unique) relative to a particular sequence of basic commutators:

\[
h = \hat{c}_{i_1}^{e_1} \cdots \hat{c}_{i_s}^{e_s},
\]

where \( \hat{c}_i \) is the image of \( c_i \) under the natural quotient map and each \( e_i \) is an integer mod \( e \). It should be noted that the sequence of basic commutators for \( H_{d,e,c} \) is finite.

From this discussion, it follows that:

**Lemma 5.1.** Let \( s \) be the number of elements of a sequence of basic commutators in \( H_{d,e,c} \). Then the order of \( H_{d,e,c} \) is at most \( e^s \).

**Definition 5.2.** A group \( G \) is a *Baer group* if every cyclic subgroup of \( G \) is subnormal in \( G \).

**Definition 5.3.** A group \( G \) is *\( p \)-abelian* if its commutator subgroup \( G' \) is a finite \( p \)-group.

It should be noted that some authors use the term \( p \)-abelian differently. We will find FC-elements and the FC center of a group to be helpful.

**Definition 5.4.** An element \( a \) of group \( G \) is called an *FC-element* if the set \( \{a^g \mid g \in G\} \) of all conjugates of \( a \) in \( G \) is finite.

**Definition 5.5.** The *FC-center* of group \( G \) is the set of all FC-elements of \( G \).

The Rips and Shalev examples make use of several ways of moving between rings, algebras, and groups.

**Definition 5.6.** Let \( R \) be a ring. A *unit* of \( R \) is an element \( r \in R \) that possesses a multiplicative inverse in \( R \). We use the notation \( U(R) \) for the multiplicative group of units of \( R \).
Definition 5.7. Let $G$ be a group and $F$ a field. The group algebra $FG$ is defined to be the set of formal sums $\sum_{g \in G} r_g g$, where $r_g \in F$ and all but finitely many $r_g = 0$. Addition is defined on $FG$ by

$$
(\sum_{g \in G} r_g g) + (\sum_{g \in G} r'_g g) = \sum_{g \in G} (r_g + r'_g) g,
$$

and multiplication is defined on $FG$ by

$$
(\sum_{g \in G} r_g g)(\sum_{g \in G} r'_g g) = \sum_{g \in G} (\sum_{xy=g} r_x r'_y) g.
$$

Furthermore, scalar multiplication of $F$ on $FG$ is given by

$$
 r(\sum_{g \in G} r_g g) = \sum_{g \in G} (rr_g) g.
$$

Definition 5.8. Let $G$ be a group and $F$ a field. The augmentation ideal of the group ring $FG$ is the subset of $FG$ of those elements $\sum_{g \in G} r_g g$ for which $\sum r_g = 0$.

We also need to define a few more concepts related to Lie rings.

Definition 5.9. Let $R$ be a Lie ring and let $x$ be an element of $R$. Then the mapping $\text{ad}_x : R \to R$ is given by $\text{ad}_x(a) = (x, a)$ for each $a$ in $R$. This map is called the inner derivation of the Lie ring $R$ at the element $x$.

Definition 5.10. Let $R$ be a Lie ring and let $x \in R$. An $n$-stable series for $\text{ad}_x$ is a decreasing sequence of Lie subrings

$$
R = R_0 \supset R_1 \supset \cdots \supset R_n = 0,
$$

where $(R_{i+1}, R_i) \subset R_{i+1}$ and where $(x, R_i) \subset R_{i+1}$. The element $x$ of a Lie ring $R$ is $n$-Baer if $R$ has a $n$-stable series for $\text{ad}_x$, and the Lie ring is Baer if every element is $n$-Baer for some $n$. If there is a positive integer $n$ such that $R$ has an $n$-stable series
for \(ad_x\) for every element \(x\) of \(R\), then \(R\) is \(n\)-Baer. A Lie ring is called \(Baer\)-bounded if it is \(n\)-Baer for some positive integer \(n\).

We also use a concept from number theory in our calculations:

**Definition 5.11.** Let \(i\) be a positive integer, with the factorization \(p_1^{e_1} \ldots p_s^{e_s}\), where \(p_1, \ldots, p_s\) are distinct primes. Then we define the Moebius function \(\mu\) by

\[
\mu(i) = \begin{cases} 
1 & \text{for } i = 1, \\
(-1)^s & \text{if } e_k = 1 \text{ for all } k \text{ in } \{1, 2, \ldots, s\}, \\
0 & \text{otherwise}. 
\end{cases}
\]

**2.2. General Lemmas.** We start by gathering several results about semidirect products. Let \(C\) and \(D\) be groups. First, observe that the inverse of the element \((c,d)\) in the semidirect product \(C \rtimes D\) is \((c^{-1},d^{-1})\).

**Lemma 5.2.** In the semidirect product \(C \rtimes D\), if \(m,n \in C\) and \(s,t \in D\), then

\[
(m,s)^{(n,t)} = ((n^{-1}mn^s)t^{-1},t^{-1}st) 
\]

and

\[
[(m,s),(n,t)] = (m^{-s-1}(n^{-1}mn^s)t^{-1}t^{-1},s^{-1}t^{-1}st). 
\]

If \(D\) is an abelian group, then

\[
(m,s)^{(n,t)} = ((n^{-1}mn^s)t^{-1},s) 
\]

and

\[
[(m,s),(n,t)] = (m^{-s-1}n^{-1}s^{-1}m^{t-1}t^{-1}n^{t-1},1). 
\]

If \(C\) and \(D\) are both abelian, then

\[
[(m,s),(n,t)] = ((m^{-1}m^{t-1})^{s-1}(n^{-s-1}n)^{t-1},1). 
\]

**Proof.** Let \(m,n \in C\) and \(s,t \in D\), so \((m,s)\) and \((n,t)\) are in \(C \rtimes D\). Then
\[(m, s)^{(n, t)} = (n, t)^{-1}(m, s)(n, t)\]
\[= (n^{-t^{-1}}, t^{-1})(mm^s, st)\]
\[= (n^{-t^{-1}}(mn^s)^{-1}, t^{-1}st)\]
\[= ((n^{-1}mn^s)^{-1}, t^{-1}st).\]  \hspace{1cm} (74)

Also,

\[[(m, s), (n, t)] = (m, s)^{-1}(m, s)^{(n, t)}\]
\[= (m^{-s^{-1}}, s^{-1})((n^{-1}mn^s)^{-1}, t^{-1}st)\]
\[= (m^{-s^{-1}}(n^{-1}mn^s)^{-1}s^{-1}, s^{-1}t^{-1}st).\] \hspace{1cm} (75)

The abelian cases now follow easily. \[\square\]

We also need a formula for exponents in a semidirect product.

**Lemma 5.3.** For any positive integer \(k\), if \((m, s)\) is an element of the semidirect product \(C \rtimes D\), then \((m, s)^k = (mm^s \ldots m^{s^{k-1}}, s^k)\).

**Proof.** The proof follows by induction on \(k\). We first notice that the base case of the induction is trivial. Now

\[(m, s)^k = (m, s)^{k-1}(m, s)\]
\[= (mm^s \ldots m^{s^{k-2}}, s^{k-1})(m, s)\]
\[= (mm^s \ldots m^{s^{k-2}}m^{s^{k-1}}, s^{k-1}s)\]
\[= (mm^s \ldots m^{s^{k-1}}, s^k).\] \hspace{1cm} (76)

We also use a basic result about solvability of quotients of Lie rings.

**Lemma 5.4.** Let \(L\) be a Lie ring and let \(M\) be an ideal of \(L\). If \(M\) is Lie-solvable and \(L/M\) is commutative, then \(L\) is also Lie-solvable.
Proof. Because $L/M$ is commutative, $L^{(1)} = [L, L]$ is a subset of $M$. Since $M$ is solvable, $L$ is also solvable. □

We will use the following properties:

**Definition 5.12.** We say that a group $G$ has property $\mathfrak{C}$ if there exists a sequence $\{\beta_1, \beta_2, \ldots\}$ of positive integers such that for every positive integer $d$, every subgroup $H$ of $G$ with $d$ generators is nilpotent of class at most $\beta_d$.

**Definition 5.13.** A group $G$ has property $\mathfrak{D}$ if there is a sequence $\{\gamma_1, \gamma_2, \ldots\}$ of positive integers such that for every positive integer $d$, for every subgroup $H$ of $G$ with $d$ generators, the order of $H$ is at most $\gamma_d$.

We will use the following results about $p$-groups:

**Lemma 5.5.** Let $G$ be a finite $p$-group. If $G$ has order $v$, then $G$ is nilpotent of class at most $\log_p(v) - 1$.

Proof. This follows by result 5.3.1 (i) in [64], which states that if $G$ is a group of order $p^{m+1}$, then $G$ is nilpotent of class at most $m$. □

**Lemma 5.6.** The number of elements in a set of basic commutators of $H_{d,e,n}$ is at most

$$\sum_{i=1}^{n} \left( \sum_{j=1}^{\lfloor n/i \rfloor} \frac{\mu(i)}{i} \cdot \frac{d^j}{j} \right).$$

(77)

Proof. Witt’s theorem ([63], page 169) states that the number of basic commutators of weight $m$ for a free group on $d$ generators is given by

$$\sum_{i \mid m} \left( \frac{\mu(i) d^{m/i}}{m} \right).$$

(78)

Because the terms of a sequence of basic commutators in $H_{d,e,n}$ are images of the terms of a sequence of basic commutators of weight at most $n$ in the free group on $d$
generators, the number of basic commutators of weight \(m\), where \(1 \leq m \leq n\), in a sequence for \(H_{d,e,n}\) is at most

\[
\sum_{i \mid m} \left( \frac{\mu(i)d^{m/i}}{m} \right).
\]  

The total number of basic commutators is the sum of the number of basic commutators of each weight up to the nilpotence class. Thus, the number of basic commutators for \(H_{d,e,n}\) of all weights is at most

\[
\sum_{m=1}^{n} \left( \sum_{i \mid m} \left( \frac{\mu(i)d^{m/i}}{m} \right) \right) = \sum_{i=1}^{n} \left( \sum_{m \leq n, i \mid m} \left( \frac{\mu(i)d^{m/i}}{m} \right) \right) 
= \sum_{i=1}^{n} \left( \sum_{j=1}^{\lfloor n/i \rfloor} \left( \frac{\mu(i)d^{j}}{j^{i}} \right) \right),
\]  

where \(j = m/i\).

**Proposition 5.7.** Let \(G\) be a \(d\)-generator group with exponent \(e\) and nilpotence class at most \(c\). Then \(G\) has order less than or equal to:

\[
e^\left( \sum_{i=1}^{c} \left( \sum_{j=1}^{\lfloor e/i \rfloor} \left( \frac{\mu(i)d^{j}}{j^{i}} \right) \right) \right).
\]  

**Proof.** We first note that \(G\) is a quotient of \(H_{d,e,c}\). If \(b\) is the number of basic commutators in a sequence of basic commutators for \(H_{d,e,c}\), then, by Lemma 5.1, the order of \(H_{d,e,c}\) is at most \(e^b\). Hence, by Lemma 5.6, \(H_{d,e,c}\) has order at most

\[
e^\left( \sum_{i=1}^{c} \left( \sum_{j=1}^{\lfloor e/i \rfloor} \left( \frac{\mu(i)d^{j}}{j^{i}} \right) \right) \right).
\]  

We may now prove the following Corollary:
Corollary 5.8. Let $G$ be a $p$-group with exponent $e$ and property $\mathfrak{C}$. Then $G$ has property $\mathfrak{D}$.

Proof. Because $G$ satisfies property $\mathfrak{C}$, there is a sequence $\{\beta_1, \beta_2, \ldots\}$ of positive integers such that for every positive integer $d$, every subgroup $H$ of $G$ with $d$ generators is nilpotent of class at most $\beta_d$. We need to show that there is a sequence $\{\gamma_1, \gamma_2, \ldots\}$ such that the order of a $d$-generator subgroup of $G$ is at most $\gamma_d$. Let $H$ be a $d$-generator subgroup of $G$. Since $G$ has property $\mathfrak{C}$, we know that $H$ has nilpotence class at most $\beta_d$. By hypothesis, $H$ is a $p$-group of exponent $e$. By Proposition 5.7, $H$ is of order at most

$$p \cdot e^{\left(\sum_{i=1}^{\beta_d} \left(\sum_{j=1}^{\beta_d/i} \left(\frac{\mu(j)}{j} \cdot \frac{e}{i}\right)\right)\right)},$$

(84)

so we may set

$$\gamma_d = p^{\left(\sum_{i=1}^{\beta_d} \left(\sum_{j=1}^{\beta_d/i} \left(\frac{\mu(j)}{j} \cdot \frac{e}{i}\right)\right)\right)}.$$  

(85)

□

Proposition 5.9. Let $G$ be a $p$-group with exponent $e$ and property $\mathfrak{C}$, with associated sequence $\{\beta_d\}$ and let $s$ be a positive integer. Let $H$ be the wreath product $H = G \wr \mathbb{Z}_p$ and let $K$ be an $s$-generator subgroup of $H$. Then $K$ has order less than or equal to:

$$p \cdot e^{\left(\sum_{i=1}^{\beta_{sp}} \left(\sum_{j=1}^{\beta_{sp}/i} \left(\frac{\mu(j)}{j} \cdot \frac{e}{i}\right)\right)\right)}.$$  

(86)

Proof. Let $\{x_1, \ldots, x_s\}$ be a set of generators of $K$, and let $B$ be the base group of the wreath product $H = G \wr \mathbb{Z}_p$. Because $G \wr \mathbb{Z}_p = B \rtimes \mathbb{Z}_p$, we can write each $x_i$ as $(b_i, z_i)$, where $b_i$ is an element of $B$ and $z_i$ is an element of $\mathbb{Z}_p$. Since $B$ is the direct product of $p$ copies of $G$, each $b_i$ can be written $(g_{i,1}, \ldots, g_{i,p})$, where $g_{i,j}$ is an element of $G$. Let $\psi_j : B \rightarrow G$ be the mapping determined by $\psi_j((g_1, \ldots, g_p)) = g_j$. Now define $L = \langle \psi_j(g_{i,j}) \mid i \in \{1, \ldots, s\}, j \in \{1, \ldots, p\}\rangle$. In other words, $L$ is the subgroup generated by the elements of $G$ that are used to construct $\{x_1, \ldots, x_s\}$. Now let $M = L \wr \mathbb{Z}_p$. We note that $L$ is a subgroup of $G$, so
\[ M \text{ is a subgroup of } H. \text{ Also, } K \text{ is a subgroup of } M. \text{ The group } L \text{ is a subgroup of } G \text{ that has at most } sp \text{ generators, and so by Proposition 5.7, the order of } L \text{ is at most } \\
\left( \sum_{i=1}^{\beta sp} \left( \sum_{j=1}^{[\beta sp/i]} \left( \frac{\mu(j) \cdot (sp)^j}{i} \right) \right) e \left( \sum_{i=1}^{\beta sp} \left( \sum_{j=1}^{[\beta sp/i]} \left( \frac{\mu(j) \cdot (sp)^j}{i} \right) \right) \right) \right). \]  

Thus the base group of \( M \) has order at most 
\[ e \left( \sum_{i=1}^{\beta sp} \left( \sum_{j=1}^{[\beta sp/i]} \left( \frac{\mu(j) \cdot (sp)^j}{i} \right) \right) \right) p, \]  

and hence \( M \) has order at most 
\[ p \cdot e \left( \sum_{i=1}^{\beta sp} \left( \sum_{j=1}^{[\beta sp/i]} \left( \frac{\mu(j) \cdot (sp)^j}{i} \right) \right) \right). \]  

Because \( K \) is a subgroup of \( M \), the order of \( K \) is also at most 
\[ p \cdot e \left( \sum_{i=1}^{\beta sp} \left( \sum_{j=1}^{[\beta sp/i]} \left( \frac{\mu(j) \cdot (sp)^j}{i} \right) \right) \right). \]  

□

Corollary 5.10 follows immediately from Proposition 5.9.

**Corollary 5.10.** Let \( G \) be a \( p \)-group of exponent \( e \) that has property \( \mathcal{C} \). Let \( H \) be the wreath product \( H = G \wr \mathbb{Z}_p \). Then \( H \) has property \( \mathcal{D} \).

**Proposition 5.11.** Let \( G \) be a \( p \)-group of exponent \( e \) that has property \( \mathcal{C} \), with associated sequence \( \{\beta_d\} \). Let \( H \) be the wreath product \( H = G \wr \mathbb{Z}_p \), and let \( K \) be an \( s \)-generator subgroup of \( H \). Then \( K \) is nilpotent of class at most:
\[ p \cdot \log_p e \cdot \left( \sum_{i=1}^{\beta sp} \left( \sum_{j=1}^{[\beta sp/i]} \left( \frac{\mu(j)}{j} \cdot \frac{(sp)^j}{i} \right) \right) \right). \]  

**Proof.** In Proposition 5.9, we showed that the order of \( K \) is at most 
\[ p \cdot e \left( \sum_{i=1}^{\beta sp} \left( \sum_{j=1}^{[\beta sp/i]} \left( \frac{\mu(j) \cdot (sp)^j}{i} \right) \right) \right). \]  

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By Lemma 5.5, the nilpotence class of $K$ is at most
\[ p \cdot \log_p e \cdot \left( \sum_{i=1}^{\beta_{sp}} \left( \sum_{j=1}^{\left\lfloor \beta_{sp}/i \right\rfloor} \left( \frac{\mu(j)}{j} \cdot \frac{(sp)^i}{i} \right) \right) \right). \tag{93} \]

The final three introductory results we use are quoted without proof.

**Lemma 5.12.** (Passi, Passman, and Seghal [65]) Let $p$ be a prime number and let $G$ be a group. If $K$ is a field of characteristic $p$, then
(a) the group ring $KG$ is Lie-nilpotent if and only if $G$ is $p$-abelian and nilpotent;
(b) if $p \neq 2$, then $KG$ is Lie-solvable if and only if $G$ is $p$-abelian;
(c) if $p = 2$, then $KG$ is Lie-solvable if and only if $G$ has a 2-abelian subgroup of index at most 2.

**Lemma 5.13.** (Bovdi and Kripta (or Hripta) [66], translated [67]) Let $p$ be a prime number. Let $K$ be a field of characteristic $p \geq 3$. Let $G$ be a periodic group and let $P$ be a Sylow $p$-subgroup of $G$. Then the group of units $U(KG)$ of the group ring $KG$ is solvable if and only if either

1. the commutator subgroup of $G$ is a finite $p$-group, or
2. $K$ is a field of 3 elements, $P$ is finite and normal in $G$, and one of the following conditions holds for $E = G/P$:
   (a) $E$ has an abelian normal subgroup $A$ of exponent 8, where $D = E/A$ has order 2 and $dad^{-1} = a^3$ for all $d \in D$ and $a \in A$.
   (b) $E$ is a 2-group of nilpotency class 2, and the nonabelian subgroups of its homomorphic images contain no central elements of order 4.

**Lemma 5.14.** (Bovdi and Kripta (or Hripta) [66], translated [67]) Let $K$ be a field of characteristic 2, and let $G$ be a periodic group. Then the group $U(KG)$ is
solvable if and only if $G$ contains a finite normal 2-subgroup $N$ such that $G_1 = G/N$ satisfies one of the following conditions:

(1) $G_1$ is abelian.

(2) $G_1$ is the direct product of an abelian group $W$ having no elements of order 2 and a 2-group $B$ such that $B$ contains an abelian normal subgroup of index 2, and the quotient group of $B$ by its center has bounded exponent.

(3) $K$ is the field of 2 elements and the group $G_1$ is an extension of the direct product $M \times W$ of an abelian 2-group $M$ of bounded exponent and an elementary abelian 3-group $W$ by the group $\langle b \rangle$ of order 2, with, moreover, $bwb^{-1} = w^{-1}$ for all $w \in W$.

2.3. Construction for an Odd Prime. In this subsection we discuss (and give many more details than the published account of) a proof of the theorem:

**Theorem 5.15.** (Rips and Shalev [6]) For every prime $p \neq 2$, there is some positive integer $n$ for which there is an $n$-Engel $p$-group that has an element whose normal closure in the whole group is not $(n - 1)$-Engel.

We prove Theorem 5.15 by constructing a sequence of groups and rings $G_p$, $R_p$, $H_p$, and $M_p$. We use the group $M_p$ to prove by contradiction that for $p$-groups, condition $\mathfrak{E}_n$ does not imply condition $\mathfrak{L}(\mathfrak{E}_{n-1})$ for some positive integer $n$.

**Construction 2 (Construction of $G_p$).** (Rips and Shalev [6]) For each prime $p \neq 2$, let

$$I_p = \langle (a_i, z_i) \mid a_i^p = z_i^p = [a_i, a_j] = [a_i, z_j] = [z_i, z_j] = 1 \text{ for } i, j \in \{1, 2, \ldots\} \rangle,$$

so that $I_p$ is $(C_p \times C_p)^\omega$. Let $X_p$ be the group $\langle x \mid x^p = 1 \rangle$ and let $G_p$ be the group $I_p \rtimes X_p$, where the action of $X_p$ on $I_p$ is defined by $a_i^x = a_i z_i$ and $z_i^x = z_i$.

**Lemma 5.16.** (Rips and Shalev, [6]) The following hold:

(1) $G_p$ is nilpotent of class 2.

(2) $G_p$ is of exponent $p$.  

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(3) $G_p$ has an abelian subgroup of index $p$.

(4) $G_p$ is not $p$-abelian.

(5) $\mathbb{Z}_p G_p$ is not Lie-solvable.

**Proof.**

(1) Because $X_p$ and $I_p$ are abelian, for $m, n \in I_p$ and $s, t \in X_p$, by Lemma 5.2,

$$[(m, s), (n, t)] = ((m^{-1} m^{t^{-1}})^{s^{-1}} (n^{-s} n)^{t^{-1}}, 1). \quad (94)$$

Now if $m = (a_v^1 z_1 a_2 z_2 \ldots)$ and $n = (a_v^1 x_1 a_2 z_2 \ldots)$, and if also $s = x^i$ and $t = x^j$, where $w_k, v_k, \tilde{w}_k, \tilde{v}_k, i, j$ are all integers, then

$$[(m, s), (n, t)] = ((m^{-1} m^{t^{-1}})^{s^{-1}} (n^{-s} n)^{t^{-1}}, 1)$$

$$= (((a_v^1 z_1 a_2 z_2 \ldots)^{-1} (a_v^1 x_1 a_2 z_2 \ldots)^{x^{-i}} (a_v^1 z_1 a_2 z_2 \ldots)^{x^{-i}})^{x^{-i}} (a_v^1 z_1 a_2 z_2 \ldots)^{x^{-i}}, 1)$$

$$= (((a_v^1 z_1 a_2 z_2 \ldots) (a_v^1 x_1 a_2 z_2 \ldots)^{x^{-i}} (a_v^1 z_1 a_2 z_2 \ldots)^{x^{-i}} (a_v^1 z_1 a_2 z_2 \ldots)^{x^{-i}}, 1)$$

$$= (((a_v^1 z_1 a_2 z_2 \ldots) (a_v^1 x_1 a_2 z_2 \ldots)^{x^{-i}} (a_v^1 z_1 a_2 z_2 \ldots)^{x^{-i}} (a_v^1 z_1 a_2 z_2 \ldots)^{x^{-i}}, 1) \quad (95)$$

$$= (((a_v^1 a_2 z_2 \ldots) (a_v^1 x_1 a_2 z_2 \ldots)^{x^{-i}} (a_v^1 a_2 z_2 \ldots)^{x^{-i}}, 1)$$

We have shown that any commutator of elements in $G_p$ is contained in the subgroup $\langle z_k \mid k \in \{1, 2, \ldots \}\rangle$, and so is in $Z(G_p)$. Hence $G_p$ is nilpotent of class 2.

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(2) We show that $G_p$ is of exponent $p$ by computing the $p$th power of an arbitrary element $((a_1^wz_1^1a_2^z2_2\ldots), x^j)$ of $G_p$, where $w_k, v_k, j$ are all integers. By Lemma 5.3 and the action of $X_p$ on $I_p$,

$$(((a_1^wz_1^1a_2^z2_2\ldots), x^j)^p = (((a_1^wz_1^1a_2^z2_2\ldots)(a_1^wz_1^1a_2^z2_2\ldots))x^j$$

$$\cdots (a_1^wz_1^1a_2^z2_2\ldots)(x^j)^{p-1}, (x^j)^p)$$

$$= (((a_1^wz_1^1a_2^z2_2\ldots)(a_1^wz_1^1+jv_1a_2^z2_2+jv_2\ldots)$$

$$\cdots (a_1^wz_1^1+(p-1)v_1)a_2^z2_2z_2^2+(p-1)v_2\ldots), (x^p)^j)$$

$$= (((a_1^wz_1^1a_2^z2_2\ldots)(a_1^wz_1^1+jv_1+1\ldots+(p-1)v_1a_2^z2_2z_2^2+jv_2\ldots), (x^p)^j)$$

$$= (((a_1^wz_1^1a_2^z2_2\ldots)(a_1^wz_1^1+jv_1+1\ldots+(p-1)v_1a_2^z2_2z_2^2+jv_2\ldots), (x^p)^j).$$

Because $p$ is an odd prime and the $a_i$, the $z_i$, and $x$ are all of order $p$, we conclude that $G_p$ is of exponent $p$.

(3) $((C_p \times C_p)^\omega, 1)$ is an abelian subgroup of $G_p$ of index $p$.

(4) In part (1) of this proof, we show that the subgroup $\langle z_i \mid i \in \{1, 2, \ldots \} \rangle$ contains $G'_p$. Each $z_i$ is in $G'_p$ because $[a_i, x] = z_i$. Hence $G'_p$ is the subgroup $\langle z_i \rangle$, and so is infinite. Hence $G_p$ is not $p$-abelian.

(5) This follows from (4) and part (b) of Lemma 5.12.

\[ \square \]

**Construction 3 (Construction of $R_p$).** *(Rips and Shalev, [6])* For each prime $p \neq 2$, define $R_p$ to be the ring

$$\mathbb{Z}_p G_p \Delta F_p,$$

where $F_p = FC(G_p)$, the subgroup of all FC-elements of $G_p$, and $\Delta F_p$ is the augmentation ideal of $\mathbb{Z}_p F_p$.

We will also need the notation (used in [68]) that $\Delta(G_p, F_p)$ is the left ideal of $\mathbb{Z}_p G_p$ generated by $\{ h - 1 \mid h \in F_p \}$. It should be noted that some authors use $\Delta$ differently.
**Lemma 5.17.** (Rips and Shalev, [6]) The following claims hold:

1. \(F_p = (I_p, 1)\), which is an elementary abelian \(p\)-group.
2. \(G_p \cong F_p \rtimes C_p\).
3. \(a^p = 0\) for every \(a \in \Delta(F_p)\).
4. \(R_p = \Delta(G_p, F_p)\).
5. \(R_p = \bigoplus_{0 \leq i < p} x^i \cdot \Delta F_p\).

**Proof.** (1) We first compute the conjugate of one arbitrary element of \(G_p\) by another. Let \(g = ((a_1^{v_1} z_1^{-w_1}, a_2^{v_2} z_2^{-w_2}, ...), x^s)\) and \(\hat{g} = ((a_1^{v_1} z_1^{-w_1}, a_2^{v_2} z_2^{-w_2}, ...), x^t)\) be elements of \(G_p\), where the \(w_i, v_i, \hat{w}_i, \hat{v}_i, s, and t\) are integers modulo \(p\).

Then, by the observation earlier that the inverse of the element \((c, d)\) in the semidirect product \(C \rtimes D\) is \((c^{-d^{-1}}, d^{-1})\) and by the action of \(X_p\) on \(I_p\),

\[
g^{\hat{g}} = ((a_1^{v_1} z_1^{-w_1}, a_2^{v_2} z_2^{-w_2}, ...), x^s)((a_1^{\hat{v}_1}, a_2^{\hat{v}_2}, ...), x^t) \]
\[
= ((a_1^{v_1} z_1^{-w_1}, a_2^{v_2} z_2^{-w_2}, ...), x^t) \cdot ((a_1^{\hat{v}_1} z_1^{\hat{w}_1}, a_2^{\hat{v}_2} z_2^{\hat{w}_2}, ...), x^s) \]
\[
= ((a_1^{v_1} z_1^{-w_1}, a_2^{v_2} z_2^{-w_2}, ...), x^{s+t})
\]

(98)
If \( s = 0 \), then there are \( p \) choices for \( t \), so \( g \) has \( p \) conjugates, but if \( s \) is not 0, then \( g \) has infinitely many conjugates because there are infinitely many choices for the sequence \( \{\hat{v}_i\} \). We conclude that \( F_p = (I_p, 1) \).

(2) We defined \( G_p \) to be \( I_p \rtimes X_p \). Since \( X_p \) is \( \mathbb{Z}_p \) and we have shown that \( I_p \cong (I_p, 1) = F_p \), we conclude that \( G_p \cong F_p \rtimes \mathbb{Z}_p \).

(3) By the definition of augmentation ideal, any element \( a \) of \( \Delta F_p \) is of the form \( r_1g_1 + r_2g_2 + \cdots + r_mg_m \), where \( r_i \in \mathbb{Z}_p \), \( g_i \in G_p \), and \( \sum r_i = 0 \). In a commutative ring over \( \mathbb{Z}_p \), we know \((i + j)^p = i^p + j^p\), so

\[
a^p = (r_1g_1 + r_2g_2 + \cdots + r_mg_m)^p = (r_1g_1)^p + (r_2g_2)^p + \cdots + (r_mg_m)^p
\]

\[
= r_1^p1_G + r_2^p1_G + \cdots + r_m^p1_G + r_{m-1}^p1_G + r_m^p1_G. 
\]

In \( \mathbb{Z}_p \), we know \( r_i^p = r_i \), so

\[
a^p = r_1^p1_G + r_2^p1_G + \cdots + r_{m-1}^p1_G + r_m^p1_G
\]

\[
= (r_1 + r_2 + \cdots + r_{m-1} + r_m)1_G
\]

\[
= 0 \cdot 1_G
\]

\[
= 0. 
\]

(4) By part (i) of exercise 2 on page 138 of [68], \( R_p = \Delta(G_p, F_p) \).

(5) We first note that \( \{x^i \mid i \in \{0, \ldots, p-1\}\} \) is a transversal of \( F_p \) in \( G_p \). By Proposition 3.3.3 of [68], \( \{x^i(h-1) \mid i \in \{0, \ldots, p-1\}, h \in F_p, h \neq 1\} \) is a basis for \( \Delta(G_p, F_p) \). However, \( \Delta(G_p, F_p) \) is defined as the left ideal of \( \mathbb{Z}_pG_p \) generated by \( \{h - 1 \mid h \in F_p\} \). This gives the desired conclusion.

\(\Box\)

For \( u \) a unit of \( \mathbb{Z}_pG_p \) and \( r \in \mathbb{Z}_pG_p \), we can use the notation \( ru^a \) to denote \( u^{-1}ru \) in \( \mathbb{Z}_pG_p \). In particular, any element of \( G_p \) is a unit in \( \mathbb{Z}_pG_p \). By part (5) of Lemma
5.17, every \( y \) in \( R_p \) may be written uniquely in the form \( \sum_{0 \leq i < p} x^i \cdot y_i \), where \( y_i \in \Delta F_p \). Let \( S_y = \{ y_{ij} : 0 \leq i, j < p \} \). Also, let \( n_p = 1 + (p - 1)p^2 \).

**Lemma 5.18.** *(Rips and Shalev [6])* Let \( y \) be an element of \( R_p \). Then the following statements hold:

(1) \( b^p = 0 \) for every element \( b \in S_y \).
(2) \( | S_y | \leq p^2 \).
(3) \( S_{y}^{n_p} = 0 \).
(4) \( S_y \cdot \mathbb{Z}_p G_p = \mathbb{Z}_p G_p \cdot S_y \).
(5) \( (S_y \cdot \mathbb{Z}_p G_p)^{n_p} = (S_y)^{n_p} \cdot \mathbb{Z}_p G_p = 0 \).
(6) \( S_y \cdot \mathbb{Z}_p G_p \) is a 2-sided ideal of \( \mathbb{Z}_p G_p \).
(7) \( y \in S_y \cdot \mathbb{Z}_p G_p \).
(8) \( (\mathbb{Z}_p G_p y \mathbb{Z}_p G_p)^{n_p} = 0 \).
(9) \( r^{n_p} = 0 \) for every \( r \) in \( R_p \).
(10) \( R_p \) is Baer-bounded.
(11) \( \mathbb{Z}_p G_p / R_p \cong \mathbb{Z}_p C_p \) as rings, where \( C_p = \langle x \rangle \).
(12) \( \mathbb{Z}_p G_p / R_p \) is a commutative ring.
(13) \( R_p \) is not a Lie-solvable ring.

**Proof.**

(1) Any \( b \in S_y \) is of the form \( y_{x}^{i} \), where \( y_i \) is in \( \Delta(F_p) \), so \( b^p = (y_{x}^{i})^p = (y_i^p)^{x^i} = 0 \) by part (4) of Lemma 5.17.

(2) \( | S_y | \leq p^2 \) because the elements of \( S_y \) are determined by making two choices, each of which has \( p \) possibilities.

(3) We want to show that any product of \( n_p \) elements of \( S_y \) is trivial. We first note that \( F_p \) is abelian, and that any conjugate of an element of \( F_p \) by \( x \) remains in \( F_p \). Thus the elements of \( S_y \) commute. Because \( b^p = 0 \) for each \( b \in S_y \), a product that contains any factor \( p \) or more times is trivial. Since \( S_y \) has at most \( p^2 \) elements, \( p^2(p - 1) \) is the largest number of factors in a nontrivial product.
(4) We want to show that \( S_y \cdot \mathbb{Z}_pG_p = \mathbb{Z}_pG_p \cdot S_y \). To do so, for \( s \) an element of \( S_y \), we show that each element of \( s\mathbb{Z}_pG_p \) is a sum of elements of the form \( t_kg_kz \), where \( t_k \in \mathbb{Z}_p \), \( g_k \in G_p \), and \( z \in S_y \). Let \( v \in s\mathbb{Z}_pG_p \), so \( v = s \sum t_kg_k \) where \( t_k \in \mathbb{Z}_p \) and \( g_k \in G_p \). Then

\[
v = s \sum t_kg_k = \sum (st_kg_k) = \sum t_k (sg_k) = \sum t_k g_k s^{q_k}.\]

We now consider one of the summands \( t_k g_k s^{q_k} \). We need to show that \( s^{q_k} \) is in \( S_y \). Because \( s \) is in \( S_y \), we can write \( s = y_i^{x^j} \), where \( y_i \in \Delta(F_p) \). Thus \( s^{q_k} = y_i^{x^j} \). We know that \( x^j g_k \in G_p \), so \( x^j g_k = f x^m \), where \( f \in F_p \) and \( m \in \{0, \ldots, p-1\} \). Then \( s^{q_k} = y_i^{f x^m} = (y_i^f)^x^m \). Now we consider \( y_i^f \). Because \( y_i \in \Delta(F_p) \), it is of the form \( r_1 g_1 + r_2 g_2 + \cdots + r_m g_m \), where \( r_i \in \mathbb{Z}_p \), \( g_i \in F_p \), and \( \sum r_i = 0 \). We calculate \( y_i^f = (r_1 g_1 + r_2 g_2 + \cdots + r_m g_m)^f = r_1 g_1^f + r_2 g_2^f + \cdots + r_m g_m^f \).

Because \( F_p \) is abelian, \( y_i^f = r_1 g_1 + r_2 g_2 + \cdots + r_m g_m = y_i \), so \( s^{q_k} = y_i^{x^m} \in S_y \).

(5) By part (4), \( (S_y \cdot \mathbb{Z}_pG_p)^{n_p} = (\mathbb{Z}_pG_p)^{n_p} \cdot (S_y)^{n_p} \cdot \mathbb{Z}_pG_p = 0 \cdot \mathbb{Z}_pG_p = 0 \).

(6) \( (S_y \cdot \mathbb{Z}_pG_p)\mathbb{Z}_pG_p \subset S_y \cdot \mathbb{Z}_pG_p \) and by part (4),

\[
\mathbb{Z}_pG_p(S_y \cdot \mathbb{Z}_pG_p) = \mathbb{Z}_pG_p(\mathbb{Z}_pG_p \cdot S_y) \subset \mathbb{Z}_pG_p \cdot S_y = S_y \cdot \mathbb{Z}_pG_p
\]

(7) Since each \( y_i^{x^j} \cdot x^i \) is in \( S_y \cdot \mathbb{Z}_pG_p \), so also is

\[
y = \sum_{0 \leq i < p} x^i \cdot y_i = \sum_{0 \leq i < p} y_i^{x^{-i}} \cdot x^i.
\]

(8) We know

\[
(\mathbb{Z}_pG_p y \mathbb{Z}_pG_p)^{n_p} \subset (\mathbb{Z}_pG_p (S_y \cdot \mathbb{Z}_pG_p) \mathbb{Z}_pG_p)^{n_p} \subset (\mathbb{Z}_pG_p S_y \cdot \mathbb{Z}_pG_p \mathbb{Z}_pG_p)^{n_p}
\]

\[
= (\mathbb{Z}_pG_p S_y \cdot \mathbb{Z}_pG_p)^{n_p}
\]

\[
= (S_y \cdot \mathbb{Z}_pG_p \mathbb{Z}_pG_p)^{n_p}
\]

\[
= (S_y \cdot \mathbb{Z}_pG_p)^{n_p}
\]

By part (5), we conclude that \( (\mathbb{Z}_pG_p y \mathbb{Z}_pG_p)^{n_p} = 0 \).

(9) This clearly follows from part (8)

(10) We need to show that there is an \( n \)-stable series for \( ad_y \). We define the series \( R_1, \ldots, R_{n_p} \) by \( R_k = (\mathbb{Z}_pG_p y \mathbb{Z}_pG_p)^k \) for \( 1 \leq k \leq n_p \). We notice for \( i > j \) that
(\(Z_p G_p y Z_p G_p\))^i (\(Z_p G_p y Z_p G_p\))^j \subseteq (\(Z_p G_p y Z_p G_p\))^{i+j} \subseteq (\(Z_p G_p y Z_p G_p\))^i.\) Hence for \(i > j\), the Lie commutator \(((\(Z_p G_p y Z_p G_p\))^i, (\(Z_p G_p y Z_p G_p\))^j)\) is a sum and difference of elements of \((\(Z_p G_p y Z_p G_p\))^i\), so

\[((\(Z_p G_p y Z_p G_p\))^i, (\(Z_p G_p y Z_p G_p\))^j) \subseteq (\(Z_p G_p y Z_p G_p\))^{i+j+1} \text{ because } y \in \(Z_p G_p y Z_p G_p\).\] However, it follows from (8) that \((\(Z_p G_p y Z_p G_p\))^n_p = 1\), so \(R_p\) is \(n_p\)-Baer.

(11) Corollary 3.3.5 of [68] states that \(Z_p G_p / R_p = Z_p G_p / \Delta(G_p, F_p) \cong Z_p (G_p / F_p) \cong Z_p C_p\), where \(C_p\) is the group of integers mod \(p\), in this case \(\langle x \rangle\).

(12) It follows from (10) and the fact that \(Z_p(x)\) is a commutative ring that \(Z_p G_p / R_p\) is commutative.

(13) We showed in part (6) of Lemma 5.16 that \(Z_p G_p\) is not Lie-solvable, but in part (11) we showed that \(Z_p G_p / R_p\) is Lie-solvable. By Lemma 5.4, \(R_p\) is not Lie-solvable.

\[\square\]

**Construction 4 (Construction of \(H_p\)).** (Rips and Shalev [6]) For each prime \(p \neq 2\), let \(H_p\) be the multiplicative group \(1 + R_p\).

**Lemma 5.19.** (Rips and Shalev [6]) The following hold:

(1) \(H_p\) is a subset of \(U(Z_p G_p)\), the group of units of \(Z_p G_p\).

(2) \(H_p\) is a subgroup of \(U(Z_p G_p)\).

(3) \(H_p\) is a \(p\)-group of exponent at most \(p^3\).

(4) For each \(h \in H_p\), the normal closure \(h^{H_p}\) of \(h\) in \(H_p\) is nilpotent of class less than \(n_p\).

(5) \(U(Z_p G_p)\) is not a solvable group.

(6) \(U(Z_p G_p) / H_p\) is an abelian group.

(7) \(H_p\) is not a solvable group.

(8) \(H_p\) is a locally finite group.

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If \( h_1, \ldots, h_d \) are elements of \( H_p \), then the subgroup of \( H_p \) generated by \( h_1, \ldots, h_d \) has the following properties:

(a) it is the product of the cyclic subgroups generated by the \( h_i \);
(b) it is normal in \( H_p \);
(c) it is nilpotent of class less than or equal to \( d(n_p - 1) \).

(10) \( H_p \) has property \( D \).

**Proof.**

1. We first note that \( R_p \subset \mathbb{Z}_p G_p \). Let \( r \in R_p \). Because

\[
(1 - (-r))(1 + (-r) + (-r)^2 + \cdots + (-r)^{n_p-1}) = 1 - (-r)^{n_p} = 1 + r^{n_p} = 1,
\]

the element \( 1 + r \) has an inverse in \( \mathbb{Z}_p G_p \), and so is a unit of \( \mathbb{Z}_p G_p \).

2. Note that the product \( (1 + r_1)(1 + r_2) = 1 + (r_1 + r_2 + r_1 r_2) \) is in \( H_p \). Also, the inverse of \( 1 + r \) shown in the proof of (1) is also in \( H_p \). Thus \( H_p \) is a subgroup of \( U(\mathbb{Z}_p G_p) \).

3. Let \( r \in R_p \), so \( 1 + r \in H_p \). Then

\[
(1 + r)^{p^3} = \sum_{i=0}^{p^3} \binom{p^3}{i} r^i
= 1 + r^{p^3}
= 1 + r^{p^3 - p^2 + 1} r^{p^2 - 1}
= 1 + r^{n_p} r^{p^2 - 1}.
\]

In part (8) of Lemma 5.18, we showed that for any \( r \in R_p \), the power \( r^{n_p} \) is trivial. Hence

\[
(1 + r)^{p^3} = 1 + 0 \cdot r^{p^2 - 1}
= 1.
\]

Therefore \( H_p \) has exponent dividing \( p^3 \).
(4) Let \( h \) be an element of \( H_p \). Consider the commutator \([k_1, k_2, \ldots, k_{n_p}]\), where each \( k_i \) is a conjugate of \( h \), that is, \( k_i = h^{h_i} \) for some \( h_i \in H_p \). Each \( h_i \) is of the form \( 1 + r_i \), where \( r_i \in R_p \), so

\[
k_i = (1 + r)^{(1 + r_i)} = (1 + r_i)^{-1}(1 + r)(1 + r_i) = (1 + r_i)^{-1}(1 + (1 + r_i)^{-1}r)(1 + r_i)
\]

\[= (1 + r_i)^{-1}(1 + r_i) + (1 + r_i)^{-1}r(1 + r_i)
= 1 + (1 + r_i)^{-1}r(1 + r_i)
= 1 + r^{(1+r_i)}.
\]

Therefore

\[
[k_1, k_2, \ldots, k_{n_p}] = [1 + r^{(1+r_1)}, 1 + r^{(1+r_2)}, \ldots, 1 + r^{(1+r_{n_p})}]. \tag{105}
\]

By Lemma 3.1,

\[
[k_1, k_2, \ldots, k_{n_p}] = 1 + (r^{(1+r_1)}, r^{(1+r_2)}, \ldots, r^{(1+r_{n_p})}) + b, \tag{106}
\]

where \( b \) is a sum of terms of higher weight and \((\quad, \quad)\) is the algebra commutator in \( R_p \). Now we examine the ring commutator \((r^{(1+r_1)}, r^{(1+r_2)}, \ldots, r^{(1+r_{n_p})})\). Because we defined \((r, s)\) to be \( rs - sr\), the commutator \([k_1, k_2, \ldots, k_{n_p}]\) is a sum of terms of the form

\[r^{(1+r_{\sigma(1)})}r^{(1+r_{\sigma(2)})} \cdots r^{(1+r_{\sigma(n_p)})},\]

where \( \sigma \in Sym_{n_p} \), the symmetric group on \( n_p \) elements. Each \( r^{(1+r_{\sigma(i)})} \) is in \( \mathbb{Z}_p G_p r \mathbb{Z}_p G_p \) because \( R_p \subset \mathbb{Z}_p G_p \). Therefore each \( r^{(1+r_{\sigma(1)})}r^{(1+r_{\sigma(2)})} \cdots r^{(1+r_{\sigma(n_p)})} \) is in \( (\mathbb{Z}_p G_p r \mathbb{Z}_p G_p)^{n_p} \). However,

\[(\mathbb{Z}_p G_p r \mathbb{Z}_p G_p)^{n_p} = 0, \]

so \( r^{(1+r_{\sigma(1)})}r^{(1+r_{\sigma(2)})} \cdots r^{(1+r_{\sigma(n_p)})} = 0 \). The same argument also shows that the higher power terms in the sum \( b \) are trivial. Therefore we conclude that \([k_1, k_2, \ldots, k_{n_p}] = 1\) and hence \( h^{H_p} \) is nilpotent of class less than \( n_p \).
(5) We showed in the proof of part (4) of Lemma 5.16 that the commutator subgroup of \( G_p \) is not finite, so for \( p > 3 \), Lemma 5.13 shows that \( U(\mathbb{Z}_p G_p) \) is not solvable. If \( p = 3 \), we need only notice that the Sylow 3-subgroup of \( G_p \) is all of \( G_p \), which is not finite.

(6) In part (11) of Lemma 5.18, we showed that \( \mathbb{Z}_p G_p / R_p \) is commutative. Hence \( U(\mathbb{Z}_p G_p)/(1 + R_p) \) is an abelian group.

(7) If \( H_p \) were solvable, then \( U(\mathbb{Z}_p G_p) \), the extension of the solvable group \( H_p \) by an abelian group, would also be solvable, which contradicts part (5).

(8) Let \( B \) be a finitely generated subgroup of \( H_p \). By Parts (3) and (4), \( B \) is a finitely generated nilpotent \( p \)-group of finite exponent. Thus, by Proposition 5.7, \( B \) is finite, so \( H_p \) is locally finite.

(9) By Fitting’s Theorem the product of \( d \) normal nilpotent subgroups of class less than or equal to \( n_p - 1 \) is a normal nilpotent subgroup of class less than or equal to \( d(n_p - 1) \).

(10) This follows immediately from Proposition 5.8 and parts (3) and (9).

\[ \square \]

**Construction 5 (Construction of \( M_p \)).** *(Rips and Shalev [6]*) For each prime \( p \neq 2 \), let \( M_p \) be the wreath product \( H_p \wr C_p \). The base group \( B \) of \( M_p = H_p \wr C_p \) is \( \Pi_{i=1}^p H_{p,i} \), where each \( H_{p,i} \) is an isomorphic copy of \( H_p \) via the natural isomorphism \( \phi_i : H_p \to H_{p,i} \). For \( h \) in \( H_p \), let \( h_i = \phi_i(h) \). The action of \( x \), a generator of \( C_p \), on \( B \) is given by \( h_i^x = h_{i+1} \) for \( 1 \leq i \leq p - 1 \) and \( h_1^x = h_1 \).

**Lemma 5.20.** *(Rips and Shalev [6]*) The following hold:

(1) \( M_p \) has property \( \mathcal{D} \).

(2) \( M_p \) is locally nilpotent.

(3) \( M_p \) is \( n \)-Engel for some positive integer \( n \).

(4) \( M_p \) is not a Baer group.
Proof. (1) Lemma 5.19 part (10) and Corollary 5.10 show that $M_p$ has property $D$.

(2) By part (1), any finitely generated subgroup of $M_p$ is finite, and since every finite $p$-group is nilpotent, it follows that $M_p$ is locally nilpotent.

(3) By part (1), $M_p$ has property $D$. Hence every 2-generator subgroup of $M_p$ has bounded order. Thus, every 2-generator subgroup has bounded nilpotence class $n$, for some positive integer $n$. If $b$ and $c$ are elements of $M_p$, the commutator $[b, n, c]$ is a commutator of weight $n + 1$ in $\langle b, c \rangle$. Thus $[b, n, c]$ is trivial, so $M_p$ is $n$-Engel.

(4) Assume that $M_p$ is a Baer group, so every cyclic subgroup of $M_p$ is subnormal in $M_p$. In particular, $\langle x \rangle$ is subnormal in $M_p$. Thus there is a decreasing sequence of subgroups $M_p = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_m = \langle x \rangle$. In the following we will at times identify $x$ with $(1, \ldots, 1)$.

We now show that $H_p^{(m)}$ is trivial. We start by showing that $((H_p', 1, \ldots, 1), 1)$ is in $N_1$. Let $h$ be in $H_p$. Now

\[
[x, ((h^{-1}, 1, \ldots, 1), 1)] = ((h^{-1}, 1, \ldots, 1), 1)^{-x}((h^{-1}, 1, \ldots, 1), 1)
\]

\[
= ((h, 1, \ldots, 1), 1)^x((h^{-1}, 1, \ldots, 1), 1)
\]

\[
= ((1, h, 1, \ldots, 1), 1)((h^{-1}, 1, \ldots, 1), 1)
\]

\[
= ((1, h, 1, \ldots, 1), 1)((h^{-1}, 1, \ldots, 1), 1)
\]

\[
= ((h^{-1}, h, 1, \ldots, 1), 1),
\]

and so $((h^{-1}, h, 1, \ldots, 1), 1) \in [M_p, x] \subseteq N_1$. Because $N_1$ is normal in $N_0$, the conjugate of an element of $N_1$ by an element of $N_0$ is in $N_1$. Thus, for $k$ in $H_p$,

\[
((h^{-1}, h, 1, \ldots, 1), 1)^{(1,k,1,\ldots,1)}
\]

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\[= ((1, k, 1, \ldots, 1), 1)^{-1}((h^{-1}, h, 1, \ldots, 1), 1)((1, k, 1, \ldots, 1), 1)\]
\[= ((h^{-1}, k^{-1}hk, 1, \ldots, 1), 1)\]
\[= ((h^{-1}, h^k, 1, \ldots, 1), 1)\]
\[\in N_1.\]

Moreover,

\[
[x, ((h^k, 1, \ldots, 1), 1)] = ((h^k, 1, \ldots, 1), 1)^{-x}((h^k, 1, \ldots, 1), 1)
= ((h^{-k}, 1, \ldots, 1), 1)^x((h^k, 1, \ldots, 1), 1)
= ((1, h^{-k}, 1, \ldots, 1), 1)((h^k, 1, \ldots, 1), 1)
= ((h^k, h^{-k}, 1, \ldots, 1), 1)
\in N_1,
\]

so

\[
((h^{-1}, h^k, 1, \ldots, 1), 1)((h^k, h^{-k}, 1, \ldots, 1), 1) = ((h^{-1}h^k, h^k h^{-k}, 1, \ldots, 1), 1)
= (\langle [h, k], 1, \ldots, 1, 1 \rangle)
\in N_1.
\]

Hence \((H'_p, 1, \ldots, 1, 1)\) is a subset of \(N_1\).

Now let \(b\) and \(c\) be elements of \(H_p^{(i-1)}\) and assume that 
\((H_p^{(i-1)}, 1, \ldots, 1, 1) \subseteq N_{i-1}\). Then \((b, 1, \ldots, 1, 1)\) is an element of \(N_{i-1}\).

Hence,

\[
[x, ((b^{-1}, 1, \ldots, 1), 1)] = ((b^{-1}, 1, \ldots, 1), 1)^{-x}((b^{-1}, 1, \ldots, 1), 1)
= ((1, b, 1, \ldots, 1), 1)((b^{-1}, 1, \ldots, 1), 1)
= ((b^{-1}, b, 1, \ldots, 1), 1),
\]

so \((b^{-1}, b, 1, \ldots, 1, 1) \in [N_{i-1}, x] \subseteq N_i\). Moreover,
\((b^{-1}, b, 1, \ldots, 1, 1)\)^{(1,c,1,\ldots,1),1} \\
= ((1, c, 1, \ldots, 1, 1)^{-1}((b^{-1}, b, 1, \ldots, 1, 1), 1)((1, c, 1, \ldots, 1, 1), 1) \\
= ((1, c^{-1}, 1, \ldots, 1, 1)^{-1}((b^{-1}, b, 1, \ldots, 1, 1), 1)((1, c, 1, \ldots, 1, 1), 1) \\
= ((b^{-1}, b^{c}, 1, \ldots, 1, 1), 1) \\
\in N_{i}.

By the induction hypothesis, \((b^{c}, 1, \ldots, 1, 1)\) is in \(N_{i-1}\). This means

\[[x, (b^{c}, 1, \ldots, 1, 1, 1)] = ((b^{c}, 1, \ldots, 1, 1, 1)^{-x}((b^{c}, 1, \ldots, 1, 1)) \\
= ((1, b^{-x}, 1, \ldots, 1, 1)((b^{c}, 1, \ldots, 1, 1), 1) \\
= ((b^{c}, b^{-x}, 1, \ldots, 1, 1), 1) \\
\in N_{i},

so

\([(b^{-1}, b^{c}, 1, \ldots, 1, 1, 1)((b^{c}, b^{-x}, 1, \ldots, 1, 1, 1), 1) = ((b^{-1}b^{c}, b^{c}b^{-x}, 1, \ldots, 1, 1), 1) \\
= ([b, c], 1, \ldots, 1, 1), 1) \\
\in N_{i}.

Therefore \(((H_{p}^{j}), 1, \ldots, 1, 1, 1)\) is a subset of \(N_{i}\). By induction, 
\(((H_{p}^{j}), 1, \ldots, 1, 1, 1) \subseteq N_{j} \) for all \(j, 1 \leq j \leq m\), and so 
\(((H_{p}^{m}), 1, \ldots, 1, 1, 1) \subseteq ((1, \ldots, 1)(x))\). Thus \(H_{p}^{(m)}\) is trivial. This contradicts the fact that \(H_{p}\) is not solvable, and hence \(M_{p}\) is not a Baer group.

\[\square\]

**Proof of Theorem 5.15.** Let \(p\) be an odd prime. We showed in part (3) of Lemma 5.20 that \(M_{p}\) is \(n\)-Engel for some positive integer \(n\). Assume for the sake of contradiction that for every positive integer \(i \leq n\), the normal closure of every element of an \(i\)-Engel \(p\)-group is \((i - 1)\)-Engel. We show that this assumption leads
to a contradiction of known properties of $M_p$. Let $m$ be an element of $M_p$. Let $N_1$ be the normal closure of $m$ in $M_p$. We then inductively define $N_s$ to be $m^{N_{s-1}}$ for $s = 2, \ldots, n$. The subgroup $N_1$ is normal in $M_p$ and, by our assumption, is $(n-1)$-Engel. If $N_{s-1}$ is $(n-s+1)$-Engel, the induction hypothesis says that $N_s$ is $(n-s)$-Engel. Thus, each $N_s$ is $(n-s)$-Engel and normal in $N_{s-1}$, so subnormal in $M_p$. Then $N_{n-1}$ is subnormal in $M_p$, and is $(n-(n-1))$-Engel, meaning that it is abelian. Hence $N_n$ is an abelian subgroup that contains $m$ and is subnormal in $M_p$. Since $\langle m \rangle$ is normal in $N_n$, we conclude that $\langle m \rangle$ is subnormal of defect $n+1$ in $M_p$. Since this argument holds for every element $m$ in $M_p$, we have proved that $M_p$ is Baer, which contradicts part (4) of Lemma 5.20.

2.4. Construction for $p = 2$. Rips and Shalev make the comment that although their construction works only for odd primes, it can be modified to work for $p = 2$ also. In this subsection we show how to make the necessary modifications to the proof.

**Theorem 5.21.** For some positive integer $n$, there is an $n$-Engel 2-group that has an element whose normal closure in the whole group is not $(n-1)$-Engel.

The construction of such a group is very similar to that of Rips and Shalev’s construction for odd primes, and in fact follows from a hint in their paper [6].

**Construction 6 (Construction of $G_2$).** Let $I_2 = \langle a_i, z_i \mid a_i^4 = z_i^4 = [a_i, a_j] = [z_i, z_j] = 1 \text{ for } i, j \in \{1, 2, \ldots \} \rangle$, so $I_2$ is $(C_4 \times C_4)^\omega$. Let $X_2$ be the group $\langle x \mid x^4 = 1 \rangle$ and let $G_2$ be the group $I_2 \rtimes X_2$, where the action of $X_2$ on $I_2$ is defined by $a_i^x = a_i z_i$ and $z_i^x = z_i$.

**Lemma 5.22.** The following hold:

1. $G_2$ is nilpotent of class 2.
2. $G_2$ is of exponent 8.
3. $G_2$ has an abelian subgroup of index 4.
(4) $G_2$ does not have a 2-abelian subgroup of index at most 2.
(5) $\mathbb{Z}_2G_2$ is not a Lie-solvable ring.

PROOF. (1) The proof of this is exactly the same as in part (1) of Lemma 5.16.

(2) We compute the eighth power of an element of $G_2$. By Lemma 5.3 and the action of $X_2$ on $I_2$, if $w_k$, $v_k$, and $j$ are all integers,

\[
((a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \ldots, x^j)^8 = ((a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \ldots)^{(x^j)^7}, (x^j)^8) \\
\ldots (a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \ldots)^{(x^j)^7}, (x^j)^8) \\
= ((a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \ldots)^{(a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \ldots)^{(1+j)v_1} a_2^{w_2}(1+j)v_2} \ldots) \\
\ldots (a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \ldots)^{(1+j)v_1} a_2^{w_2}(1+j)v_2) \\
= ((a_1^{8w_1} z_1^{(8+4j(7))v_1} a_2^{8w_2}(8+4j(7)v_2) \ldots, 1) \\
= ((a_1^{8w_1} z_1^{(8+4j(7))v_1} a_2^{8w_2}(8+4j(7)v_2) \ldots, 1).
\]

Because the $a_i$ and the $z_i$ are of order 4, we conclude that $G_2$ is of exponent at most 8. Because $(a_1z_1, x^4) = (a_1^4z_1^{10}, x^4) = (z_1^2, 1)$ is not trivial, the exponent of $G_2$ is greater than 4, and so is exactly 8.

(3) The subgroup $((C_4 \times C_4)^{\omega}, 1)$ is of index 4 in $G_2$ and is abelian.

(4) In the proof of part (1) of Lemma 5.16, we computed the commutator of any two elements of $G_p$ for odd prime $p$. The same calculation holds in $G_2$. Thus $\langle z_i \rangle$ contains $G'_2$. Each $(z_i, 1)$ is in $G'_2$ because $[a_i, x] = z_i$. Hence $G'_2$ is the subgroup $\langle z_i \rangle$ of $G_2$, and so is infinite. Therefore $G_2$ is not 2-abelian.

Assume that $W$ is a 2-abelian subgroup of index 2 in $G_2$. Because $G_2$ is a 2-group, $W$ is a subgroup of index 2 in $G_2$ with finite commutator subgroup $W'$. We see that $(1, x^2) \in W$ because $(1, x^2) = (1, x)^2$. Suppose for a positive integer $i$ that $(a_i, x) \in W$. Then $[(a_i, 1), (1, x^2)] = (z_i^2, 1)$ is also in $W$. The $\{z_k\}$ are generators of a relatively free abelian group of exponent 4, so $(z_i^2, 1) = (z_j^2, 1)$ only when $i = j$. Thus, because $W'$ is finite, we deduce
that there are only finitely many $i$ with $(a_i, 1)$ in $W$. Next, we consider a left coset of $W$. If $(a_i, 1)$ and $(a_j, 1)$ are both in the same left coset of $W$, then $(a_i, 1)^{-1}(a_j, 1) = (a_i^{-1}a_j, 1) \in W$. However, this conclusion implies that $[(a_i^{-1}a_j, 1), (1, x^2)] = (z_i^j z_j^i, 1)$ is also in $W$. Thus only finitely many $(a_i, 1)$ are in any one left coset of $W$. Because there are infinitely many $a_i$ and only finitely many are in $W$, we conclude that the index of $W$ in $G$ is infinite, which is a contradiction, so no such $W$ exists.

(5) This follows immediately from part (c) of Lemma 5.12.

\[\Box\]

**Construction 7** (Construction of $R_2$). We define $R_2$ to be the ring

\[\mathbb{Z}_2 G_2 \Delta F_2,\]  

where $F_2 = FC(G_2)$ and $\Delta F_2$ is the augmentation ideal of $\mathbb{Z}_2 F_2$.

As in the odd prime case, $\Delta(G_2, F_2)$ is the left ideal of $\mathbb{Z}_2 G_2$ generated by

\[\{h - 1 \mid h \in F_2\} \] .

**Lemma 5.23.** The following hold:

(1) $F_2$ is abelian.
(2) $G_2 \cong F_2 \rtimes C_4$.
(3) $a^8 = 0$ for every $a \in \Delta(F_2)$.
(4) $R_2 = \Delta(G_2, F_2)$.
(5) $R_2 = \bigoplus_{0 \leq i < 4} x^i \cdot \Delta F_2$.

**Proof.**

(1) The proof that $F_2 = ((C_4 \times C_4)^{\omega}, 1) = (I_2, 1)$ is exactly the same as for part (1) of Lemma 5.17.

(2) The proof is identical to the proof of part (2) of Lemma 5.17.

(3) By the definition of augmentation ideal, any element $a$ of $\Delta F_2$ is of the form $r_1 g_1 + r_2 g_2 + \cdots + r_m g_m$, where $r_i \in \mathbb{Z}_2$, $g_i \in G_2$, and $\sum r_i = 1$. In a ring over $\mathbb{Z}_2$, $(i + j)^p = i^8 + j^8$, so
\[ a^8 = (r_1g_1 + r_2g_2 + \cdots + r_{m-1}g_{m-1} + r_mg_m)^8 \]
\[ = (r_1g_1)^8 + (r_2g_2)^8 + \cdots + (r_{m-1}g_{m-1})^8 + (r_mg_m)^8 \]
\[ = r_1^81_G + r_2^81_G + \cdots + r_{m-1}^81_G + r_m^81_G \]
\[ = (r_1 + r_2 + \cdots + r_{m-1} + r_m)1_G. \]

Since \( \sum r_i = 0 \), we conclude that \( a^8 = 0 \).

(4) By part (i) of exercise 2 on page 138 of [68], \( R_2 = \Delta(G_2, F_2) \).

(5) We first note that \( \{ x^i | i \in \{0, \ldots, 3\} \} \) is a transversal of \( F_2 \) in \( G_2 \). By Proposition 3.3.3 of [68], \( \{ x^i(h-1) | i \in \{0, \ldots, p-1\}, h \in F_2, h \neq 1 \} \) is a basis for \( \Delta(G_2, F_2) \). However, \( \Delta(G_2, F_2) \) is defined as the left ideal of \( \mathbb{Z}_2G_2 \) generated by \( \{ h - 1 | h \in F_2 \} \). This gives the desired conclusion.

By part (5) of Lemma 5.23, every \( y \in R_2 \) may be written uniquely in the form \( \sum_{0 \leq i < 4} x^i \cdot y_i \), where \( y_i \in \Delta F_2 \). Let \( S_y = \{ y_i^{x^i} : 0 \leq i, j < 4 \} \). Also let \( n_2 = 7 \cdot 16 + 1 = 113 \).

**Lemma 5.24.** Let \( y \in R_2 \). Then the following hold:

1. \( b^8 = 0 \) for every element \( b \in S_y \).
2. \( | S_y | \leq 16 \).
3. \( S_y^{n_2} = 0 \).
4. \( S_y \cdot \mathbb{Z}_2G_2 = \mathbb{Z}_2G_2 \cdot S_y \).
5. \( (S_y \cdot \mathbb{Z}_2G_2)^{n_2} = (S_2)^{n_2} \cdot \mathbb{Z}_2G_2 = 0 \).
6. \( S_y \cdot \mathbb{Z}_2G_2 \) is a 2-sided ideal of \( \mathbb{Z}_2G_2 \).
7. \( y \in S_y \cdot \mathbb{Z}_2G_2 \).
8. \( (\mathbb{Z}_2G_2y\mathbb{Z}_2G_2)^{n_2} = 0 \).
9. \( r^{n_2} = 0 \) for every \( r \) in \( R_2 \).
10. \( R_2 \) is Baer-bounded.

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(11) $\mathbb{Z}_2G_2/R_2 \cong \mathbb{Z}_2(x)$ as rings.
(12) $\mathbb{Z}_2G_2/R_2$ is a commutative ring.
(13) $R_2$ is not a Lie-solvable ring.

**Proof.**

(1) Any $b \in S_y$ is of the form $y_i^x$ where $y_i$ is in $\Delta(F_2)$, so
\[ b^8 = (y_i^x)^8 = (y_i^8)^x = 0. \]

(2) The elements of $S_y$ are made by making two choices, each of which has four possibilities.

(3) We want to show that any product of $n_2$ elements of $S_y$ is trivial. We first note that $F_2$ is abelian, and that any conjugate of an element of $F_2$ by $x$ remains in $F_2$. Thus the elements of $S_y$ commute. Because $b^8 = 0$ for each $b \in S_y$, a product that contains any factor eight or more times is trivial. Since $S_y$ has at most 16 elements, this gives 16(7) as the largest number of factors in a nontrivial product.

(7) Since each $y_i^x \cdot x^i$ is in $S_y \cdot Z_2G_2$, so also is
\[ y = \sum_{0 \leq i < 4} x^i \cdot y_i = \sum_{0 \leq i < 4} y_i^{-i} \cdot x^i. \]

The proofs of parts (4)-(6) and (8)-(13) are the same as the respective proofs in Lemma 5.18.

**Construction 8 (Construction of $H_2$).** Let $H_2$ be the multiplicative group $1 + R_2$.

**Lemma 5.25.** The following hold:

(1) $H_2$ is a subset of $U(Z_2G_2)$, the group of units of $Z_2G_2$.
(2) $H_2$ is a subgroup of $U(Z_2G_2)$.
(3) $H_2$ is a 2-group of exponent at most 128.
(4) For each $h \in H_2$, the normal closure $h^{H_2}$ of $h$ in $H_2$ is nilpotent of class less than $n_2$.
(5) $U(Z_2G_2)$ is not a solvable group.
(6) $U(Z_2G_2)/H_2$ is an abelian group.

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H₂ is not a solvable group.

H₂ is a locally finite group.

If h₁, . . . , h_d are elements of H₂, then the subgroup of H₂ generated by h₁, . . . , h_d has the following properties:
   (a) it is the product of the cyclic subgroups generated by the h_i;
   (b) it is normal in H₂;
   (c) it is nilpotent of class less than or equal to d(n₂ − 1).

H₂ has property D.

Proof.  
(1) We first notice that R₂ ⊂ Z₂G₂. Let r ∈ R₂. Because
(1 − (−r))(1 + (−r) + (−r)^2 + · · · + (−r)^n₂−1) = 1 − (−r)^n₂ = 1 + r^n₂ = 1,
the element 1 + r has an inverse in Z₂G₂, and so is a unit of Z₂G₂.

(2) The proof of this result is the same as the proof for part (2) of Lemma 5.19.

(3) Let r ∈ R₂, so 1 + r is in H₂. Then

\[
(1 + r)^{128} = \sum_{i=0}^{128} \left( \begin{array}{c} 128 \\ i \end{array} \right) r^i \\
= 1 + r^{128} \\
= 1 + r^{128−7·16−1+r·7·16−1} \\
= 1 + r^{n₂}·7·16−1 \\
= 1 + 0·7·16−1 \\
= 1.
\]

Hence H₂ has exponent dividing 128.

(4) The proof of this result is the same as the proof for part (4) of Lemma 5.19.

(5) Assume that U(Z₂G₂) is solvable. Then because G₂ is a 2-group, by
Lemma 5.14, G₂ contains a finite normal subgroup N such that either G/N is abelian or else G/N contains an abelian normal subgroup of index 2 and (G/N)/Z(G/N) has bounded exponent. We first ask what elements of G₂ can be in N. Because X₂ and I₂ are abelian, for m, n ∈ I₂ and s, t ∈ X₂, by
Lemma 5.2,

$$(m, s)^{(n, t)} = ((n^{-1} m)^{t^{-1}}, s). \quad (119)$$

Now if $m = (a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots)$ and $n = (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)$, and if also

$s = x^i$ and $t = x^j$, then

$$(a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots, x^i)((a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots, x^j)) = ((a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{-1} \cdot (a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots) \cdot (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{-j} \cdot (x^i)^{x^j}, x^i)$$

$$= (((a_1^{\hat{n}_1} z_1^{-w_1} a_2^{-n_2} z_2^{-w_2} \ldots) \cdot (a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots) \cdot (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{-j} \cdot (x^i)^{x^j})$$

$$= (((a_1^{n_1^{-1}} z_1^{w_1} a_2^{-n_2} z_2^{-w_2} \ldots) \cdot (a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots) \cdot (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{-j} \cdot (x^i)^{x^j})$$

$$= (((a_1^{n_1^{-1}} z_1^{-w_1} a_2^{-n_2} z_2^{-w_2} \ldots) \cdot (a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots) \cdot (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{-j} \cdot (x^i)^{x^j})$$

$$= (((a_1^{n_1^{-1}} z_1^{-w_1} a_2^{-n_2} z_2^{-w_2} \ldots) \cdot (a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots) \cdot (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{-j} \cdot (x^i)^{x^j})$$

If $i$ is not zero, then $m$ has infinitely many conjugates. Because $N$ is normal and finite, we conclude that $i$ is zero for all $m$ in $H_2$, so $N$ must be a finite subgroup of $I_2$. We know $I_2$ is abelian, and $G_2$ is not abelian, so $G_2/N$ cannot be abelian. Thus $G_2/N$ contains an abelian normal subgroup $B$ of index 2. Let $\phi : G \to G/N$ be the natural quotient homomorphism.

Any two elements of $B$ are elements of $G_2/N$, so they can be assumed to be $\phi(m, s) = \phi((a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots, x^i))$ and

$\phi(n) = \phi((a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots, x^j))$. Then using Lemma 5.2,

$$[\phi(m), \phi(n)] = \phi([(m, s), (n, t)])$$

$$= \phi((m^{-1} m)^{t^{-1}} (n^{-1} n)^{t^{-1}}, 1)$$

$$= \phi((a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots)^{-1} (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{-j} x^{-i})$$

$$= \phi((a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots)^{-j} (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{x^{-j}})$$

$$= \phi((a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots)^{-j} (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{x^{-j}})$$

$$= \phi((a_1^{n_1} z_1^{w_1} a_2^{n_2} z_2^{w_2} \ldots)^{-j} (a_1^{\hat{n}_1} z_1^{\hat{w}_1} a_2^{\hat{n}_2} z_2^{\hat{w}_2} \ldots)^{x^{-j}})$$

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\[
\begin{align*}
&= \phi((a_1^{-v_1}z_1^{-w_1}a_2^{-v_2}z_2^{-w_2} \ldots) (a_1^{-x^{-i}v_1}a_2^{-x^{-i}v_2}z_2^{-x^{-i}w_2} \ldots))x^{-i} \\
&= \phi((a_1^{-v_1}z_1^{-w_1}a_2^{-v_2}z_2^{-w_2} \ldots)(a_1^{-x^{-i}v_1}a_2^{-x^{-i}v_2}z_2^{-x^{-i}w_2} \ldots))x^{-i} \\
&= \phi((a_1^{-v_1}a_2^{-v_2} \ldots)(a_1^{-x^{-i}v_1}a_2^{-x^{-i}v_2} \ldots))x^{-i} \\
&= \phi((a_1^{-v_1}a_2^{-v_2} \ldots)(a_1^{-x^{-i}v_1}a_2^{-x^{-i}v_2} \ldots))x^{-i} \\
&= \phi(((z_1^{-jv_1}z_2^{-jv_2} \ldots)(z_i^{-jv_1}z_2^{-jv_2} \ldots), 1) \\
&= \phi(((z_1^{-jv_1}z_2^{-jv_2} \ldots)(z_i^{-jv_1}z_2^{-jv_2} \ldots), 1).
\end{align*}
\]

We know that \( B \) is abelian, so either \( j \) is zero, or else \( \phi(z_i) \) is trivial for every \( i \). However, \( N \) is finite, so \( \phi(z_i) \) can only be trivial for finitely many \( i \).

We conclude that every element of \( B \) is of the form \( \phi((m, 1)) \). However, \( \langle x \rangle \) is of order 4 in \( G_2 \), so \( \langle \phi(x) \rangle \) is of order 4 in \( G_2/N \). This contradicts our assertion that the index of \( B \) in \( G_2/N \) is 2. We conclude that \( U(Z_2G_2) \) is not solvable.

The proofs of the remaining parts are the same as the proofs of parts (6) to (10) of Lemma 5.19. \( \square \)

**Construction 9 (Construction of \( M_2 \)).** Let \( M_2 \) be the wreath product \( H_2 \wr C_2 \). The base group \( B \) of \( M_2 = H_2 \wr C_2 \) is \( \Pi_{i=1}^2 H_{2,i} \), where each \( H_{2,i} \) is an isomorphic copy of \( H_2 \) with the natural isomorphism \( \phi_i : H_2 \to H_{2,i} \). For \( h \) in \( H_2 \), let \( h_i = \phi_i(h) \).

**Lemma 5.26.** The following claims hold.

1. \( M_2 \) has property \( \mathcal{D} \).
2. \( M_2 \) is locally nilpotent.
3. \( M_2 \) is \( n \)-Engel for some positive integer \( n \).
(4) \( M_2 \) is not a Baer group.

PROOF. The proof is identical to that for Lemma 5.20. \( \square \)

PROOF OF THEOREM 5.21. The proof of Theorem 5.15 also applies here. \( \square \)

3. Bounds

3.1. Bounds for Odd Primes. In this subsection we determine for each prime \( p \) an upper bound for the least \( n \) for which the normal closure of an element of an \( n \)-Engel group need not be \((n - 1)\)-Engel. We do so by showing how to get such a bound for \( p \)-groups for each odd prime \( p \).

PROPOSITION 5.27. For each prime \( p > 2 \) there is an \( n \)-Engel \( p \)-group whose normal closure is not \((n - 1)\)-Engel for a value of \( n \) that is bounded above by:

\[
3p \sum_{k=1}^{2(p-1)p^3} \left( \sum_{d=1}^{[2(p-1)p^3/k]} \left( \frac{\mu(d)}{d} \cdot \left( \frac{2p^k}{k} \right) \right) \right). \tag{122}
\]

PROOF. If \( n \) is an upper bound for the nilpotence class of a 2-generator subgroup of \( M_p \), then we know that \( M_p \) is \( n \)-Engel since the Engel law is defined on two group elements. Thus we apply Proposition 5.11 with \( s = 2 \). The exponent of \( H_p \) is \( p^3 \) and by Lemma 5.19 part (9c), a \( 2p \)-generator subgroup of \( H_p \) is nilpotent of class \( 2(p-1)p^3 \), so \( \beta_2 = 2(p-1)p^3 \) and \( e = p^3 \). \( \square \)

The results of calculating the bound of Proposition 5.27 (using Maple) for specific primes are the following: For the prime \( p = 3 \), the bound is about \( 1 \times 10^{83} \), for \( p = 5 \) the bound is about \( 2 \times 10^{988} \), and for \( p = 7 \) the calculated bound is approximately \( 2 \times 10^{4715} \). It would be of interest to know if these bounds can be improved. An indication of how much these bounds may overestimate the actual value comes from comparing the bound of about \( 1 \times 10^{83} \) for a 3-group with the actual value of 5 for a 3-group from Section 1 of this chapter.
### 3.2. Bounds for 2

We find a bound for 2-groups using the same methodology as in the previous subsection.

**Proposition 5.28.** There is an $n$-Engel 2-group whose normal closure is not $(n - 1)$-Engel for a value of $n$ that is bounded above by:

$$
14 \sum_{k=1}^{448} \left( \sum_{d=1}^{\lfloor 448/k \rfloor} \left( \frac{\mu(d)}{d} \cdot \frac{(4)^k}{k} \right) \cdot \left( \frac{4}{d} \right)^i \right).
$$

(123)

**Proof.** If $n$ is an upper bound on the nilpotence class of a 2-generator subgroup of $M_2$, then we know that $M_2$ is $n$-Engel. Thus, we apply Proposition 5.11 with $s = 2$. The exponent of $H_2$ is 128 and by Lemma 5.25 part (9c), a $d$-generator subgroup of $H_2$ is nilpotent of class $d(n_2 - 1) = d(7 \cdot 16)$, so $\beta_d = d(7 \cdot 16)$ and $e = 128$. Thus $n$ is bounded above by

$$
2 \left( \sum_{i=1}^{f(4)} \left( \sum_{j=1}^{\lfloor f(4)/i \rfloor} \left( \frac{\mu(j)}{j} \cdot \frac{(4)^i}{i} \right) \right) \right) \cdot \log_2 128,
$$

(124)

which simplifies to

$$
2 \cdot 7 \left( \sum_{i=1}^{4 \cdot 7 \cdot 16} \left( \sum_{j=1}^{\lfloor 4 \cdot 7 \cdot 16/i \rfloor} \left( \frac{\mu(j)}{j} \cdot \frac{(4)^i}{i} \right) \right) \right),
$$

(125)

or

$$
14 \left( \sum_{i=1}^{448} \left( \sum_{j=1}^{\lfloor 448/i \rfloor} \left( \frac{\mu(j)}{j} \cdot \frac{(4)^i}{i} \right) \right) \right).
$$

(126)

The value of this bound, according to Maple, is about $2 \times 10^{268}$.
Chapter 6

A Class of Groups In Which $\mathcal{E}_n$ and $\mathcal{L}(\mathcal{E}_{n-1})$ are Equivalent

1. Preliminaries

In this chapter we consider a class of solvable groups in which condition $\mathcal{E}_n$, the $n$-Engel condition, implies condition $\mathcal{L}(\mathcal{E}_{n-1})$, that is, that the normal closure of every element is $(n - 1)$-Engel. The results of this chapter generalize Theorem 1.4. We first need a few definitions.

**Definition 6.1.** If $a$ is an element of a group $G$ and $n$ is a positive integer, then we say that $a$ is a *left $n$-Engel element* of $G$ if $[g, na] = 1$ for every $g \in G$.

**Definition 6.2.** Let $W = \{\theta_a \mid a \in A\}$ be a set of words in variables $x_1, x_2, \ldots$. Let $\mathfrak{A}$ be the class of all groups $G$ such that each $\theta_a$ reduces to the identity element when the variables $x_i$ are replaced by arbitrary elements of $G$. Then $\mathfrak{A}$ is called the *variety determined by the set of words* $W$. Each $\theta_a$ in $W$ is referred to as a *law* of the variety $\mathfrak{A}$.

Two varieties of groups are said to be *equivalent* if they contain the same groups.

We start by defining a variety of groups. For an integer $n \geq 4$, we consider a group $G$ to be a $\mathfrak{W}_n$ group if for every $i$ in $\{3, \ldots, n-1\}$ the law $[x_1, \ldots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \ldots, x_n] = 1$ holds for all $x_1, \ldots, x_n$ in $G$. In [7], Morse proved the results of this section for $\mathfrak{W}_6$ groups and $\mathfrak{W}_7$ groups. We show that there exists a positive integer $n$ such that the variety $\mathfrak{W}_{n+1}$ is not equivalent to the variety $\mathfrak{N}_n$ of groups that are nilpotent of class $n$.

**Lemma 6.1.** The variety $\mathfrak{W}_8$ is not equivalent to the variety $\mathfrak{N}_7$. 

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Proof. If we use the GAP script:

```
RequirePackage("anupq");
f:=FreeGroup(2);
ids:=[function(a,b,c,d,e,f,i,j) return LeftNormedComm(
    [a,b,LeftNormedComm([c,d]),e,f,i,j]); end,
    function(a,b,c,d,e,f,i,j) return LeftNormedComm(
    [a,b,c,LeftNormedComm([d,e]),f,i,j]); end,
    function(a,b,c,d,e,f,i,j) return LeftNormedComm(
    [a,b,c,d,LeftNormedComm([e,f]),i,j]); end,
    function(a,b,c,d,e,f,i,j) return LeftNormedComm(
    [a,b,c,d,e,LeftNormedComm([f,i]),j]); end,
    function(a,b,c,d,e,f,i,j) return LeftNormedComm(
    [a,b,c,d,e,f,LeftNormedComm([i,j])]); end];

g:=Pq(f: Prime:=11, Exponent:=11, ClassBound:=8,
    Identities:=ids);
```

then GAP returns:

```
#I Class 1 with 2 generators.
#I Class 2 with 3 generators.
#I Class 3 with 5 generators.
#I Class 4 with 8 generators.
#I Class 5 with 14 generators.
#I Class 6 with 23 generators.
#I Class 7 with 41 generators.
#I Class 8 with 48 generators.
```

Here, GAP tells us that the largest 11-group on 2 generators in the variety \( \mathfrak{W}_8 \) is not nilpotent of class 7, because it has class 8 pc-generators. From this we see that \( \mathfrak{W}_8 \) is not equivalent to \( \mathfrak{N}_7 \). □

Lemma 6.2. The variety of groups defined by the set of laws
\[
\{[x_1, \ldots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \ldots, x_n] = 1 \mid i \in \{1, \ldots, n-1\}\}
\]
is equivalent to \( \mathfrak{N}_{n-1} \).

Proof. If a group \( G \) is nilpotent of class \( n-1 \), it is clear that \( G \) satisfies
\[
[x_1, \ldots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \ldots, x_n] = 1 \text{ for every } i \in \{1, \ldots, n-1\}.
\]
However, the law for \( i = 1 \) implies nilpotence of class at most \( n-1 \), which completes the proof. □

We need a preliminary result.
**Lemma 6.3.** Let $G$ be a group and let $a$ be an element of $G$. If there is some integer $j$ such that $[a^{g_1}, a^{g_2}, \ldots, a^{g_j}, a] = 1$ for any $g_1, \ldots, g_j \in G$, then $a^G$ is nilpotent of class at most $j$.

**Proof.** It is sufficient to show that $[a^{h_1}, a^{h_2}, \ldots, a^{h_j}, a^{h_{j+1}}] = 1$, where $h_i \in G$ for $i \in \{1, 2, \ldots, j+1\}$. However,

$$[a^{h_1}, a^{h_2}, \ldots, a^{h_j}, a^{h_{j+1}}] = [a^{h_1h_{j+1}^{-1}}, a^{h_2h_{j+1}^{-1}}, \ldots, a^{h_jh_{j+1}^{-1}}, a^{h_{j+1}}], \quad (127)$$

which is trivial by hypothesis. □

**2. Properties of Class $\mathfrak{W}_m$**

**Lemma 6.4.** Let $G$ be a $\mathfrak{W}_{t+3}$ group, where $t$ is a positive integer. Let $j$ be a positive integer. Suppose $u \in \gamma_{t-j}(G)$ if $j < t$ or $u \in G$ if $j \geq t$. Also let $v \in G'$. If $a_i \in G$ for $1 \leq i \leq j$, then there exists an element $w$ of $G'$ such that $[uv, a_1, \ldots, a_j] = [u, a_1, \ldots, a_j]^w[v, a_1, \ldots, a_j]$.

**Proof.** This proof is a generalization of the proofs of Lemmas 2 and 2' in [7]. We prove this lemma by induction on $j$. By Lemma 1.1, $[uv, a_1] = [u, a_1]^w[v, a_1]$, so the result is true for $j = 1$. Now we assume that the result is true for $j \geq 1$ and we wish to show that it is true for $j + 1$. Let $u \in \gamma_{t-(j+1)}$ if $j + 1 < t$ or $u \in G$ if $j + 1 \geq t$, and let $v \in G'$. Now by Lemma 1.1,

$$[uv, a_1, \ldots, a_{j+1}] = [[u, a_1][u, a_1, v][v, a_1], a_2, \ldots, a_{j+1}]. \quad (128)$$

Note that $[u, a_1] \in \gamma_{t-j}(G)$ if $j + 1 < t$ or $[u, a_1] \in G'$ if $j + 1 \geq t$. In particular, if $j = t - 1$, then $[u, a_1] \in G' = \gamma_2(G) \subseteq \gamma_1(G) = \gamma_{t-(t-1)}(G)$. Thus $[u, a_1] \in \gamma_{t-j}(G)$ if $j < t$ or $[u, a_1] \in G'$ if $j \geq t$. Then by the induction hypothesis, there exists $w \in G'$ such that

$$[uv, a_1, \ldots, a_{j+1}] = [[u, a_1][u, a_1, v], a_2, \ldots, a_{j+1}]^w[v, a_1, a_2, \ldots, a_{j+1}]. \quad (129)$$
Using the induction hypothesis again, there is an element $\hat{w} \in G'$ such that

$$[uv, a_1, \ldots, a_{j+1}] = [u, a_1, v, a_2, \ldots, a_{j+1}]^{\hat{w}} [v, a_1, a_2, \ldots, a_{j+1}].$$  

(130)

Now we consider $[u, a_1, v, a_2, \ldots, a_{j+1}]$. We know $[u, a_1]$ is a product of simple commutators of length at least $t - j$ (or length 2 if $j \geq t$). By repeated application of Lemma 1.1, because $G$ is a $W_{t+3}$ group, and because $v \in G'$,

$$[u, a_1, v, a_2, \ldots, a_{j+1}] = 1.$$  

This completes the induction, and so completes the proof.

□

**Lemma 6.5.** Let $G$ be a $W_{t+3}$ group, where $t$ is a positive integer, and let $j$ be a positive integer. Suppose that $u \in \gamma_{t-1}(G)$ if $j < t$, or suppose that $u \in G$ if $j \geq t$. Also, suppose that $a_i \in G$ for $1 \leq i \leq j$. Then there exists an element $w \in G'$ such that $[u^{-1}, a_1, \ldots, a_j] = [u, a_1, \ldots, a_j]^{-w}$.  

**Proof.** This proof also follows the proofs of Lemmas 2 and 2’ in [7]. By Lemma 6.4 there exists an element $w$ in $G'$ such that

$$1 = [u^{-1}u, a_1, \ldots, a_j] = [u^{-1}, a_1, \ldots, a_j]^{w}[u, a_1, \ldots, a_j],$$  

and so

$$[u^{-1}, a_1, \ldots, a_j] = ([u, a_1, \ldots, a_j]^{w-1})^{-1} = [u, a_1, \ldots, a_j]^{-w^{-1}}.$$  

□

**Lemma 6.6.** Let $G$ be a $W_{t+3}$ group, where $t$ is a positive integer. Suppose $n$ is a positive integer greater than $t$. If $a$ is a left $(n + 1)$-Engel element of $G$, then $a^G$ is nilpotent of class at most $n$.  

**Proof.** This proof also follows the proofs of Lemmas 3 and 3’ in [7]. We first prove by induction on $k$, for $1 \leq k \leq n$, that $[a^{g_1}, \ldots, a^{g_k}, a_{n+1-k} a] = 1$ for all $g_1, \ldots, g_k$ in $G$. For $k = 1$ we notice that

$$[a^g, a] = [a[a, g], a].$$  

(131)

Because $n > t$, we can apply Lemma 6.4 to see that for some $w \in G'$,

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\[
[a^g, n a] = [a, n a] w [a, g] n a \\
= [a, g, n a] \\
= [(g, a)^{-1}, n a].
\]

Clearly \([g, a] \in G\). By Lemma 6.5, we conclude that \([a^g, n a] \) is equal to \([g, n+1 a]^w\) for some \(w \in G'\), but \([g, n+1 a]\) is trivial since \(a\) is a left \((n+1)\)-Engel element of \(G\).

To continue with the induction, we assume for \(2 \leq k \leq n\) that

\[
[a^{g_1}, \ldots, a^{g_{k-1}}, n+1-\gamma_{k-1} a] = 1 \quad \text{for all } g_1, \ldots, g_{k-1} \in G. \quad \text{(132)}
\]

For simplicity, let \(v = [a^{g_1}, \ldots, a^{g_{k-1}}]\). Now \(v \in \gamma_{k-1}(G)\), and \(t - (n + 1 - k) = (t - n) + k - 1 < k - 1\), so \(v \in \gamma_{(t-n)+k-1}(G)\). Then by Lemma 1.1 and repeated use of Lemma 6.4, again using the fact that \(n > t\), there exist \(w_1, w_2 \in G'\) such that

\[
[a^{g_1}, \ldots, a^{g_{k}}, n+1-k a] = [v, a^{g_{k}}, n+1-k a] \\
= [v, a [a, g_{k}], n+1-k a] \\
= [[v, [a, g_{k}]] [v, a] [v, a, [a, g_{k}]], n+1-k a] \\
= [v, [a, g_{k}], n+1-k a]^{w_1} \cdot [v, a, [a, g_{k}], n+1-k a]^{w_2} \\
= [v, [a, g_{k}], n+1-k a]^{w_1} \cdot [v, n+2-k a]^{w_2} \\
\cdot [v, a, [a, g_{k}], n+1-k a].
\]

Now we examine each commutator in the expansion of \([v, a^{g_{k}}, n+1-k a]\). Since \([v, n+2-k a] \) is trivial by the hypothesis that \(G\) is a \(\mathfrak{M}_{t+3}\) group. Hence \([a^{g_1}, \ldots, a^{g_{k}}, n+1-k a] = [v, [a, g_{k}], n+1-k a]^{w_1}\). If \(3 \leq k \leq n\), then \([v, [a, g_{k}], n+1-k a]\) is trivial because \(G\) is a \(\mathfrak{M}_{t+3}\) group. If \(k = 2\), then \(v = a^{g_1}\), so by Lemma 6.4 and Lemma 6.5, there exist \(w, w_1, w_2 \in G'\) such that
\[ [a^{g_1}, [a, g_2], n-1a] = [a[a, g_1], [a, g_2], n-1a] \\
= [a, [a, g_2], n-1a]^w \cdot [[a, g_1], [a, g_2], n-1a] \\
= [[[g_2, a]^{-1}, a]^{-1}, n-1a)^w \cdot [a, g_1, [a, g_2], n-1a] \quad (134) \]
\[ = [g_2, a, a, n-1a]^{-w_1w_2} \cdot [a, g_1, [a, g_2], n-1a] \\
= [g_2, a, n+1a]^{-w_1w_2} \cdot [a, g_1, [a, g_2], n-1a]. \]

However, \([g_2, n+1a] = 1\) because \(a\) is a left \((n + 1)-\)Engel element of \(G\), and \([a, g_1, [a, g_2], n-1a]\) is trivial because \(G\) is a \(W_{t+3}\) group.

We have shown that \([a^{g_1}, \ldots, a^{g_k}, n+1-k]a]\) = 1, which completes the induction. Thus \([a^{g_1}, \ldots, a^{g_n}, a]\) = 1 for \(k = n\) which, by Lemma 6.3, completes the proof.

\[ \square \]

The following theorem, which is the major result of this chapter, is a generalization of Theorem 1.4 ([7]).

**Theorem 6.7.** Let \(m\) be an integer greater than or equal to 4. If \(G\) is a \(W_m\) group, then for \(n \geq m - 2\), the following are equivalent:

1. \(x^G\) is nilpotent of class at most \(n\) for every \(x \in G\).
2. \(x^G\) is \(n\)-Engel for every \(x \in G\).
3. \(G\) is \((n + 1)\)-Engel.

**Proof.** We know (1) implies (2) and (2) implies (3) by Lemma 1.3. If \(G\) is an \((n + 1)\)-Engel group, then every \(x \in G\) is a left \((n + 1)\)-Engel element, so Lemma 6.6 then shows that \(x^G\) is nilpotent of class at most \(n\) for every \(x \in G\). \[ \square \]

In Chapter 3, we showed that Condition \(E_n\) and Condition \(L(\mathcal{N}_{n-1})\) are not equivalent, and in Chapter 5, we showed that Condition \(E_n\) and Condition \(L(E_{n-1})\) are not equivalent. However, in this chapter we showed that these conditions are equivalent in the case of \(W_m\) groups if \(m\) is at most \(n + 2\). As we ask in Questions 6
and 7, it is unknown if Condition $\mathcal{B}_{n-1}$ is also equivalent to Conditions $\mathcal{E}_n$, $\mathcal{L}(\mathcal{N}_{n-1})$, and $\mathcal{L}(\mathcal{E}_{n-1})$ for $\mathcal{M}_m$ groups if $m$ is at most $n + 2$. 
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Vita Auctoris

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