COMMITATOR CONDITIONS IN NORMAL CLOSURES OF ENGEL GROUPS

Christine Bussman, B.S.

An Abstract Presented to the Graduate Faculty of
Saint Louis University in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy

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Abstract

In this dissertation we explore connections between n-Engel groups and commutator properties of the normal closures of their elements. We explore examples where a group is n-Engel but an element of the group has a normal closure that is not nilpotent of class n. We also look at n-Engel groups where an element has a normal closure that is not $(n-1)-$Engel. Finally, we define and investigate a class of groups in which any n-Engel group contains only elements whose normal closures are $(n-1)-$Engel and nilpotent of class $n-1$.

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Dedication

To Mark, Andrew, and Megan

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Chapter 1

Introduction and Preliminaries

1. Summary of Results

In this dissertation, we explore how the n-Engel condition for a group relates to the $(n-1)$-Engel condition for the normal closure of every element of a group. We also see how these conditions relate to nilpotence of the normal closure of every element. The n-Engel condition and the normal closure of a group element are defined below. We first establish some standard notation.

Definition 1.1. If a and b are elements of a group G, then we define the *conjugate* of a by b, denoted by a^b, to be $b^{-1}ab$.

We will occasionally use the notation a^{-b} as an abbreviation for $(a^{-1})^b$.

Definition 1.2. If a and b are elements of a group G, then we define the *commutator* of a and b, which we denote as $[a, b]$, to be $a^{-1}b^{-1}ab$.

In this dissertation, commutators are left normed, meaning the notation $[a, b, c]$ refers to the commutator $[[a, b], c]$. We also write $[a, {}_2b]$ for the commutator $[a, b, b]$ and for n a positive integer greater than 2, we use the recursive notation $[a, {}_nb]$ for the commutator $[a, {}_{n-1}b, b]$.

Definition 1.3. For any group G, any X a subset of G, and any positive integer n, a *simple commutator* of weight n in X is a commutator of the form $[x_1, \ldots, x_n]$, where each x_i is in X.

If G is a group and n is an integer, we use the notation G^n for the subgroup $\langle g^n \mid g \in G\rangle$. Also, if H and K are subgroups of G, we use $[H, K]$ to denote the subgroup $\langle [h, k] \mid h \in H, k \in K\rangle$.

DEFINITION 1.4. Let G be a group. The *normal closure* in G of an element g of G, denoted by g^G, is defined as $\langle g^h \mid h \in G \rangle$.

DEFINITION 1.5. We define the *lower central series* of a group G to be the sequence $\gamma_1(G) \supseteq \gamma_2(G) \supseteq \ldots$ where $\gamma_1(G) = G$ and $\gamma_i(G) = [\gamma_{i-1}(G), G]$.

DEFINITION 1.6. A group G is *nilpotent* when for some positive integer c the subgroup $\gamma_{c+1}(G)$ is trivial. In this case we call the smallest such c the *nilpotence class* of G.

DEFINITION 1.7. A group G is *locally nilpotent* when every finitely generated subgroup of G is nilpotent.

DEFINITION 1.8. A group G is called an *Engel group* when for every $a, b \in G$, there is some positive integer n such that $[a, {}_n b] = 1$. If there is one positive integer n such that $[a, {}_n b] = 1$ for all pairs $a, b \in G$, then G is an *n-Engel group.*

We frequently need to calculate with commutators of group elements. The following lemma is necessary for these calculations. These identities are common knowledge, and are given without proof.

LEMMA 1.1. *Let G be a group and let a, b, and c be elements of G. Then*

(1) $[ab, c] = [a, c]^b [b, c] = [a, c][a, c, b][b, c]$.
(2) $[a, bc] = [a, c][a, b]^c = [a, c][a, b][a, b, c]$.
(3) $[a, b^{-1}] = [a, b]^{-b^{-1}} = [b, a]^{b^{-1}}$.
(4) $[a^{-1}, b] = [a, b]^{-a^{-1}} = [b, a]^{a^{-1}}$.

We also will use the following lemma, which we will prove, although it is also common knowledge.

LEMMA 1.2. *Let G be a group of nilpotence class c. If x is an element of $\gamma_{c-1}(g)$ and g and h are elements of G, then $[x, g^h] = [x, g]$.*

PROOF. By Lemma 1.1,

$$\begin{aligned}
[x, g^h] &= [x, gh][x, h^{-1}][x, h^{-1}, gh] \\
&= [x, gh][x, h^{-1}] \\
&= [x, h][x, g][x, g, h][x, h^{-1}] \\
&= [x, h][x, g][x, h^{-1}] \\
&= [x, g][x, h][x, h^{-1}] \\
&= [x, g][x, h^{-1}h][x, h^{-1}, h]^{-1} \\
&= [x, g].
\end{aligned} \tag{1}$$

□

DEFINITION 1.9. If $\mathfrak{X}$ is a class of groups, then the class of groups $\mathfrak{L}(\mathfrak{X})$ is defined to be those groups G in which the normal closure of every element of G is in the class $\mathfrak{X}$. The property of being in the class $\mathfrak{L}(\mathfrak{X})$ is called the *Levi property* generated by $\mathfrak{X}$.

Let n be a positive integer. This dissertation explores how the following conditions relate:

$\mathfrak{E}_n$: A group G has the property $\mathfrak{E}_n$ if G is n-Engel.

$\mathfrak{L}(\mathfrak{E}_n)$: A group G has the property $\mathfrak{L}(\mathfrak{E}_n)$ if the normal closure of each element a of G is n-Engel.

$\mathfrak{L}(\mathfrak{N})$: A group G has the property $\mathfrak{L}(\mathfrak{N})$ if the normal closure a^G of each element a of G is nilpotent. A group that satisfies this property $\mathfrak{L}(\mathfrak{N})$ is often called a Fitting group.

$\mathfrak{L}(\mathfrak{N}_n)$: A group G has the property $\mathfrak{L}(\mathfrak{N}_n)$ if the normal closure a^G of each element a of G is nilpotent of class at most n. Such a group is sometimes referred to as having Fitting degree n.

$\mathfrak{B}_n$: A group G has the property $\mathfrak{B}_n$ if every subgroup that is maximal in the set of subgroups of G that are nilpotent of class at most n is also normal in G. (Note that $\mathfrak{B}_n$ is used by some authors to denote a Burnside group.)

By abuse of notation, we will use the notations $\mathfrak{E}_n$, $\mathfrak{L}(\mathfrak{E}_n)$, $\mathfrak{L}(\mathfrak{N})$, $\mathfrak{L}(\mathfrak{N}_n)$, and $\mathfrak{B}_n$ to refer to both the conditions described and the classes of groups that satisfy the respective conditions.

We begin by proving simple implications among some of these conditions.

Lemma 1.3. *Let n be a positive integer. Then*

(1) *If G is a group that satisfies condition $\mathfrak{B}_n$, then G also satisfies condition $\mathfrak{L}(\mathfrak{N}_n)$.*

(2) *If G is a group that satisfies condition $\mathfrak{L}(\mathfrak{N}_n)$, then G also satisfies condition $\mathfrak{L}(\mathfrak{E}_n)$.*

(3) *If G is a group that satisfies condition $\mathfrak{L}(\mathfrak{E}_n)$, then G also satisfies condition $\mathfrak{E}_{n+1}$.*

Proof. (1) Suppose that G is a group that satisfies $\mathfrak{B}_n$. Let g be an element of the group G. Let $\mathfrak{S}$ be the set of subgroups of G that contain g and are nilpotent of class at most n. The class of groups $\mathfrak{S}$ is not empty because it contains $\langle g \rangle$. Let N be a maximal element of G. By hypothesis, N is normal in G. Thus $g^G \subset N$. Hence g^G is nilpotent of class at most n, so G also satisfies $\mathfrak{L}(\mathfrak{N}_n)$.

(2) Suppose that G is a group that satisfies $\mathfrak{L}(\mathfrak{N}_n)$. Let g be an element of G and let x and y be elements of g^G. The n-Engel commutator $[x, {}_n y]$ is a commutator of length $n+1$ in g^G. Because g^G is nilpotent of class n, we see that $[x, {}_n y]$ is trivial and so g^G is n-Engel. Thus, G also satisfies condition $\mathfrak{L}(\mathfrak{E}_n)$.

(3) Let G be a group that satisfies condition $\mathfrak{L}(\mathfrak{E}_n)$. If a and b are elements of G, then $[a, {}_{n+1} b] = [[a, b], {}_n b]$. Because $[a, b]$ is in b^G, the commutator $[a, {}_{n+1} b]$ is trivial, so G is $(n+1)$-Engel.

$\square$

Lemma 1.3 shows that for any positive integer n, condition $\mathfrak{B}_n$ implies condition $\mathfrak{L}(\mathfrak{N}_n)$, which implies condition $\mathfrak{L}(\mathfrak{E}_n)$, which in turn implies condition $\mathfrak{E}_{n+1}$. As we see in more detail in our discussion of the history of Engel groups in Section 3 of Chapter 1, it is known that conditions $\mathfrak{B}_1$, $\mathfrak{L}(\mathfrak{N}_1)$, $\mathfrak{L}(\mathfrak{E}_1)$, and $\mathfrak{E}_2$ are equivalent because $\mathfrak{E}_2$ implies $\mathfrak{B}_1$ [**1**]. Similarly, $\mathfrak{B}_2$ is stronger than $\mathfrak{L}(\mathfrak{N}_2)$, $\mathfrak{L}(\mathfrak{E}_2)$, and $\mathfrak{E}_3$, but $\mathfrak{L}(\mathfrak{N}_2)$, $\mathfrak{L}(\mathfrak{E}_2)$, and $\mathfrak{E}_3$ are equivalent because condition $\mathfrak{E}_3$ implies $\mathfrak{L}(\mathfrak{N}_2)$ [**2**]. It is known that condition $\mathfrak{L}(\mathfrak{N}_3)$ is stronger than condition $\mathfrak{E}_4$ [**3**]. It is also known that condition $\mathfrak{E}_4$ is equivalent to condition $\mathfrak{L}(\mathfrak{E}_3)$, but that condition $\mathfrak{E}_5$ is not equivalent to condition $\mathfrak{L}(\mathfrak{E}_4)$ [**4**]. Also, $\mathfrak{E}_5$ does not imply $\mathfrak{L}(\mathfrak{N})$ [**3**]. We will look at these situations in more detail in Sections 3 and 4 of Chapter 1

We explore the connections between condition $\mathfrak{L}(\mathfrak{N}_n)$ and condition $\mathfrak{E}_{n+1}$ in Chapter 3. In Chapter 3, Section 1 we explore the details of an example by Gupta and Levin [**3**]. Their example shows the following theorem.

THEOREM 3.5. *(Gupta and Levin [**3**]) There exists a 4-Engel 5-group H with an element a whose normal closure a^H is not nilpotent of class 3.*

We also show how this example can be investigated using GAP, and obtain the fact that this example is of order at most 5^{12}.

In Section 2 of Chapter 3, we find some limitations on groups that satisfy condition $\mathfrak{E}_4$ but not condition $\mathfrak{L}(\mathfrak{N}_3)$. The following theorems summarize our results.

THEOREM 3.8. *If G is a 4-Engel group and is nilpotent of class 5, then the normal closure in G of every element of G is nilpotent of class at most 3.*

THEOREM 3.9. *If G is a 4-Engel group on two generators with exponent 5, then for every element a in G, the normal closure a^G is nilpotent of class at most 3.*

Chapter 3, Section 3 discusses an example, published by Nickel [**5**], of a 4-Engel 2-group that has a normal closure that is not nilpotent of class 3. Section 4 of

Chapter 3 gives a new example of a 4-Engel 2-group that has a normal closure which is not nilpotent of class 3 and that is smaller than Nickel's example.

In Section 5 of Chapter 3, we give new examples that show that condition $\mathfrak{E}_{n+1}$ is not equivalent to condition $\mathfrak{L}(\mathfrak{N}_n)$ for other values of n besides 4, and also that 2 and 5 are not the only primes for which p-groups satisfying condition $\mathfrak{E}_{n+1}$ need not satisfy condition $\mathfrak{L}(\mathfrak{N}_n)$. We summarize our results in the following two theorems.

Theorem 3.11. *There exist 5-Engel and 6-Engel 5-groups that do not satisfy conditions $\mathfrak{L}(\mathfrak{N}_4)$ and $\mathfrak{L}(\mathfrak{N}_5)$ respectively. There also exist 5-Engel and 6-Engel 2-groups that do not satisfy conditions $\mathfrak{L}(\mathfrak{N}_4)$ and $\mathfrak{L}(\mathfrak{N}_5)$ respectively.*

Theorem 3.12. *There exist 5-Engel 3-groups and 7-groups that do not satisfy condition $\mathfrak{L}(\mathfrak{N}_4)$.*

In Chapter 4 we examine another example by Gupta and Levin from [**3**]. This example shows that condition $\mathfrak{E}_{\mathsf{n}}$ does not always imply condition $\mathfrak{L}(\mathfrak{N})$. It proves the following theorem:

Theorem 4.2. *(Gupta and Levin [**3**]) For every odd prime p, there exists a p-group that is $(p+2)$-Engel, but that has an element whose normal closure is not nilpotent.*

In Chapter 5 we consider how conditions $\mathfrak{L}(\mathfrak{E}_n)$ and $\mathfrak{E}_{n+1}$ are related. In Section 1 of 5, we consider an example by Vaughan-Lee [**4**] of a 3-group that satisfies condition $\mathfrak{E}_5$ but not $\mathfrak{L}(\mathfrak{E}_4)$. We also discuss a smaller 3-group with the same properties.

In Section 2 of Chapter 5, we examine the details of an example published by Rips and Shalev [**6**], which demonstrates the following theorem.

Theorem 5.15. *(Rips and Shalev [**6**]) For every prime $p \neq 2$, there is some positive integer n for which there is an n-Engel p-group that has an element whose normal closure in the whole group is not $(n-1)$-Engel.*

In their paper, Rips and Shalev comment that a similar construction is possible for the prime 2. In Chapter 5, Section 2.4 we work through the construction of such a 2-group, resulting in the following theorem.

THEOREM 5.21. *For some positive integer n, there is an n-Engel 2-group that has an element whose normal closure in the whole group is not $(n-1)$-Engel.*

In Chapter 5, Section 3 we calculate upper bounds for the smallest values of n in Theorem 5.15 and Theorem 5.21.

Morse [**7**] proves the following:

THEOREM 1.4. *(Morse)*

(1) *(Theorem 2 in* [**7**]*) Let $n \geq 3$ be a natural number and let G be a group of nilpotency class $n+2$. Then G is $(n+1)$-Engel if and only if the normal closure in G of every element of G is nilpotent of class n or less.*
(2) *(Theorem 3 in* [**7**]*) Let $n \geq 1$ be a natural number and let G be a group that satisfies the law $[x_1, x_2, x_3, [x_4, x_5], x_6] = 1$ for all $x_i \in G, i = 1, \ldots, 6$. Then G is $(n+1)$-Engel if and only if the normal closure in G of every element of G is nilpotent of class n or less.*
(3) *(Theorem 4 in* [**7**]*) Let $n \geq 4$ be a natural number and let G be a group that satisfies the laws $[x_1, x_2, x_3, x_4, [x_5, x_6], x_7] = 1$, $[x_1, x_2, x_3, [x_4, x_5], x_6, x_7] = 1$, and $[x_1, x_2, [x_3, x_4], x_5, x_6, x_7] = 1$ for all $x_i \in G, i = 1, \ldots, 7$. Then G is $(n+1)$-Engel if and only if the normal closure in G of every element of G is nilpotent of class n or less.*

In Chapter 6 we generalize Theorem 1.4 to find a class of groups, which we will call $\mathfrak{W}_m$, in which conditions $\mathfrak{L}(\mathfrak{N}_n)$ and $\mathfrak{E}_{n+1}$ are equivalent. For a positive integer n, we consider a group to be a $\mathfrak{W}_n$ group if for every i in $\{3, \ldots, n-1\}$ the group satisfies the law $[x_1, \ldots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \ldots, x_n] = 1$ for all $x_i \in G$.

THEOREM 6.7. *Let m be an integer greater than or equal to 4. If G is a $\mathfrak{W}_m$ group, then for $n \geq m-2$, the following are equivalent:*

(1) x^G *is nilpotent of class at most* n *for every* $x \in G$.

(2) x^G *is* n*-Engel for every* $x \in G$.

(3) G *is* $(n+1)$*-Engel.*

2. Definitions

In this section, we collect some definitions that we will use.

Definition 1.10. If G and H are two groups and $\alpha : H \to Aut(G)$ is a homomorphism from H into $Aut(G)$, we define the *semidirect product* $G \rtimes H$ to be the set $\{(g,h) \mid g \in G, h \in H\}$ with the binary operation
$(g_1, h_1)(g_2, h_2) = (g_1\alpha(h_1)(g_2), h_1h_2)$.

In the semidirect product $G \rtimes H$, the subgroup $\{(g,1) \mid g \in G\}$ is normal and isomorphic to G. Sometimes, by abuse of notation, we will consider $\{(g,1) \mid g \in G\}$ to be G. We will usually write the homomorphism action as conjugation, in which case the product is written as $(g_1, h_1)(g_2, h_2) = (g_1 g_2^{h_1}, h_1h_2)$. We will also use the fact that G is normal in $G \rtimes H$. The homomorphism α is sometimes explicitly referenced by the notation $G \rtimes_\alpha H$, but we sometimes do not specify the automorphism in the notation. The semidirect product of two groups is again a group.

We next define the wreath product of two groups G and H.

Definition 1.11. Let G and H be groups. We define the base group B of the wreath product by

$$B = \prod_{h \in H} G_h, \tag{2}$$

where each G_h is an isomorphic copy of G. We use the notation g_h, where $g \in G$ and $h \in H$, to represent the copy of g that is present in G_h. Consider the homomorphism $\alpha : H \to Aut(B)$ defined componentwise by $\alpha(h^*)(g_h) = g_{hh^*}$ for each h^* in H. Define the *wreath product* $G \wr H$ to be the semidirect product $B \rtimes_\alpha H$.

DEFINITION 1.12. A subgroup H of a group G is called *subnormal* in G if there exists a series of subgroups $H = J_0 \lhd J_1 \lhd \cdots \lhd J_n = G$. Such a series of subgroups is called a *subnormal series* for H, and is said to have length n. We say H is *subnormal of defect m* when H is subnormal in G and m is the length of the shortest subnormal series for H.

DEFINITION 1.13. The *derived series* of group G is a recursively defined series of subgroups of G that are defined as follows. Let $G^{(0)} = G$ and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for $i \in \{1, 2, \ldots\}$. A group G is *solvable* if there is some positive integer n such that $G^{(n)} = 1$. If n is the least such integer, then G is *solvable of derived length n*.

DEFINITION 1.14. A group G is a *periodic group*, also called a *torsion group*, if every element of G has finite order.

DEFINITION 1.15. A group G is *torsion free* if G does not contain any nontrivial elements of finite order.

DEFINITION 1.16. A group G is *residually finite* if for every non-trivial element g of G there is a normal subgroup H_g of finite index in G that does not contain g.

DEFINITION 1.17. If p is an odd prime and G is a p-group, then G is called a *powerful p-group* if $[G, G]$ is contained in G^p. If G is a 2-group, then G is a *powerful* 2-group if $[G, G]$ is contained in G^4.

DEFINITION 1.18. A group G is *compact* if it is considered as a topological group, and its topology is compact.

DEFINITION 1.19. A group G is *right orderable* if there exists a total order relation $\geq$ on G such that if a, b, g are elements of G, then $a \geq b$ implies $ga \geq gb$.

DEFINITION 1.20. An *associative algebra* over a field F is a set A with two binary operations $+$ and $\cdot$, referred to as addition and multiplication, and an action of F on A such that

(1) A with respect to $+$ is an abelian group;

(2) A is a semigroup with respect to multiplication;

(3) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(a+b) \cdot c = (a \cdot c) + (b \cdot c)$ for every $a, b, c \in A$;

(4) A is a vector space over F; and

(5) the action of F on A is compatible with multiplication in A in the sense that $f(a \cdot b) = (fa) \cdot b = a \cdot (fb)$ for $f \in F$ and $a, b \in A$.

Definition 1.21. A *Lie ring* R is an abelian group with an operator $(-, -)$ such that

(1) $(a+b, c) = (a, c) + (b, c)$ and $(a, b+c) = (a, b) + (b, c)$ for every $a, b, c \in R$;

(2) $(b, b) = 0$ for every $b \in R$; and

(3) $((a, b), c) + ((b, c), a) + ((c, a), b) = 0$ for every $a, b, c \in R$.

Definition 1.22. Let R be a Lie ring. We define the *derived series* of R to be the sequence of subrings $R^{(i)}$ for i in $\{0, 1, 2, \dots\}$, where $R^{(0)} = R$ and $R^{(1)} = (R, R)$ and $R^{(i)} = (R^{(i-1)}, R^{(i-1)})$ for $i > 1$. We say the Lie ring R is *Lie-solvable* if $R^{(i)} = 0$ for some positive integer i.

Definition 1.23. A *Lie algebra* L over a field F is a Lie ring over F that satisfies the additional properties $(fa + gb, c) = f(a, c) + g(b, c)$ and $(a, fb + gc) = f(a, b) + g(a, c)$ for every $a, b, c \in L$ and for every $f, g \in F$.

Definition 1.24. For a Lie algebra L, the operator $(-, -)$ is referred to as a *Lie bracket*.

In this dissertation, the Lie bracket is left normed, so (a, b, c) is used to represent $((a, b), c)$. Further, we recursively define the Lie bracket $(a, {}_n b)$ as $(a, {}_{n-1} b, b)$, where $(a, {}_1 b)$ is defined as (a, b). Also, if M and N are subalgebras of a Lie algebra L, we use the notation (M, N) to represent the subalgebra generated by the set $\{(m, n) \mid m \in M, n \in N\}$.

The Lie algebras we consider in this dissertation are almost all related to an associative algebra.

DEFINITION 1.25. Any associative algebra A has a *related Lie algebra* L. The Lie algebra has the same elements as A, with the Lie bracket given by $(a, b) = ab - ba$. In this case, the Lie bracket is called the *Lie commutator.*

DEFINITION 1.26. Let L be a Lie algebra. We define the *derived series* of L to be the sequence of subalgebras $L^{(i)}$ for i in $\{0, 1, 2, \dots\}$, where $L^{(0)} = L$ and $L^{(1)} = (L, L)$ and $L^{(i)} = (L^{(i-1)}, L^{(i-1)})$ for $i > 1$. We say the Lie algebra L is *solvable* if $L^{(i)} = 0$ for some positive integer i.

DEFINITION 1.27. Let L be a Lie algebra. We define the *lower central series* of L to be the sequence of subalgebras L^i for i in $\{1, 2, \dots\}$ given by $L^1 = L$, $L^2 = (L, L)$ and $L^i = (L^{i-1}, L)$ for $i > 2$. We say the Lie algebra L is *nilpotent* if $L^i = 0$ for some positive integer i.

DEFINITION 1.28. Let L be a Lie algebra. If for every $a, b \in L$ there is a positive integer n such that $(a, {}_n b)$ is trivial, then L is an *Engel Lie algebra.* If there is some positive integer n such that $(a, {}_n b) = 0$ for every $a, b \in L$, then L is an *n-Engel Lie algebra.*

3. History of Engel Groups

In this section we examine the history of Engel groups in general. In the next section, we consider the history of n-Engel groups in particular.

The groups now called Engel groups have become an area of interest in their own right, but their history is not well known beyond the fact that they are "related" to Lie algebras.

The earliest known reference to groups that are later called Engel groups occurs in the work of Burnside in 1902 [**8**]. Burnside's paper refers to "groups where any two conjugate operations commute," which Lemma 1.5 shows is equivalent to being a 2-Engel group.

Lemma 1.5. *A group G is a 2-Engel group if and only if any two conjugate elements of G commute.*

Proof. Any two conjugate elements commute in group G exactly when for any $a, b \in G$, we know $[a^b, a] = 1$. However,

$$\begin{aligned} [a^b, a] &= [a[a,b], a] \\ &= [a, b, a] \\ &= [[b, a]^{-1}, a] \\ &= [b, a, a]^{-[b,a]^{-1}}. \end{aligned} \tag{3}$$

Thus $[a^b, a]$ is trivial exactly when $[b, a, a]$ is trivial. □

Also in 1902, Fite [**9**] defined "class" in a way that is equivalent to the usual definition of nilpotence class, although nilpotence in groups was not defined until the 1930s.

In 1909, Fite [**10**] showed that 2-Engel groups (using terminology similar to that of Burnside) are nilpotent of class at most 3. By providing an example of a 2-Engel group of class exactly 3, Fite showed that 3 is the best possible bound on the nilpotence class of a 2-Engel group.

It seems that current interest in Engel groups comes from questions about Lie algebras. The first result of note in this direction is an unpublished result of van Kampen that finite dimensional Engel Lie algebras over fields of characteristic 0 are nilpotent. This result is referenced in Jacobson [**11**], who says that it was "found by Dr. van Kampen, Hamburg, 1928, but has not been previously published."

In 1929 Hopkins [**12**] also showed that finite 2-Engel groups are nilpotent of class at most 3. He refers to the work of Burnside [**8**], but not the work of Fite [**10**]. His terminology is also of interest because he refers to 2-Engel groups as groups where "conjugate *elements* commute," which is closer to our current usage than that of Burnside.

The same result as that of van Kampen was also proved by Weyl in class lectures (also cited by [**11**]). Jacobson [**11**] proved that finitely generated Engel Lie algebras over fields of characteristic 0 are nilpotent.

Bouton [**13**] states that Magnus and Zassenhaus drew the connection between Lie algebras and groups in the 1930s. Papers Magnus published during this time frame that include this connection are [**14**], [**15**], and [**16**]. Zassenhaus also used this connection in his 1940 paper [**17**], in which he shows that finite Engel groups are nilpotent and in which he connects Lie algebras with finite groups. On page 4 of [**17**], Zassenhaus takes credit for the label "nilpotent" for groups. Zassenhaus also lists and proves some analogous conditions for groups and Lie algebras.

In the abstracts [**18**] for an AMS conference on June 18, 1936, M. Zorn describes two talks that are of interest. In one of these ([**19**]), he proves that any finite Engel group is nilpotent, and in the other [**20**], that a finite Engel Lie algebra over a field of characteristic 0 is nilpotent (he refers erroneously to this result as related to a theorem of Lie, rather than of Engel, as he points out later [**21**]). In Zorn's work the connections between Lie algebras and groups are also apparent, because the concepts of "Engel" and "nilpotent" are used for groups, although they were first used for rings and algebras. Zorn explicitly says in the abstract about groups ([**19**]) that the group abstract is the analogue of the abstract about algebras ([**20**]). The next year, he published another paper [**21**] in which he shows that n-Engel Lie rings with the maximal condition on subrings are nilpotent. He comments that this paper replaces one of the abstracts from the previous year.

In 1940, Baer [**22**] provided examples of non-nilpotent n-Engel groups. Baer does not explicitly state that his examples have these properties, but he does state that he provides for each prime p an infinite p-group that has trivial center but abelian commutator subgroup. These groups (Example 3.4 in [**22**]) are obtained by manipulating p-adic expansions of integers, but as Gupta points out ([**23**], page 32) they are equivalent to wreath products $C_p \wr (\oplus_{i=1}^{\infty} C_p)$. Cohn [**24**] notes that Baer's examples are n-Engel but not nilpotent.

In 1942, Levi [**25**] proved the following:

THEOREM 1.6. *Let G be a 2-Engel group. Then*

(1) *G is nilpotent of class ≤ 3.*

(2) *If a, b, and c are elements of G, then $[a, b, c]^3 = 1$.*

(3) *$[a, b, c] = [b, c, a]$ for all a, b, c elements of G.*

The first part of Theorem 1.6 was already shown for finite groups in [**12**], but it is Levi's result that is usually cited now. Meier-Wunderli [**26**] proved these same results in 1949 and pointed to Fite's example of a 2-Engel group of class exactly 3 to show that 3 is the best possible bound on the nilpotence class of a 2-Engel group.

In 1953 Gruenberg [**27**] showed that a finitely generated solvable Engel group is nilpotent, and that a finitely generated solvable Engel Lie ring is nilpotent. Gruenberg's paper is of particular note because it is the earliest paper the author of this dissertation can locate in which the term "Engel group" is used. M. Zorn and others before this discuss Engel groups, but do not use the term "Engel group." Cohn [**24**] gives Gruenberg credit for first using the term "Engel group," but does not specify where he does so.

In 1954 Cohn [**24**] provided examples of a non-nilpotent Engel group and a non-nilpotent Engel ring. He credits Gruenberg with an unpublished example of such a group (his exact wording is "Dr. Gruenberg has informed me that he also has such an example"), and adds a note that after writing his paper he was made aware of Baer's 1940 example. Cohn starts with F a field of characteristic p, where p is a prime, and then constructs K, a purely transcendental extension of F of countable (infinite) transcendence degree. He then constructs M, a free left K-module on an infinite basis, and considers K to be the free K-module on the generator 1, where 1 is the unit element of K. The associative algebra A is then constructed as a sum $K+M$, and Cohn imposes a Lie multiplication on A, getting a Lie algebra L. He shows that L is $(p+1)$-Engel, but not nilpotent. Cohn's group example G is generated by elements of the form $\alpha(1+x)$, where α is a nonzero element of K and x is an element of M. This group is isomorphic to $C_p \wr (\oplus_{i=1}^{\infty} Cp)$. The group G is $(p+1)$-Engel but not nilpotent.

In 1957, Baer [**28**] defined a group to be Noetherian if every subgroup is finitely generated, and proved that Engel groups that are Noetherian are also nilpotent.

Golod [**29**] (translated [**30**]) constructed a p-group that is finitely generated, residually finite, and not locally nilpotent. He does so by considering the multiplicative group generated by $\{1+x_i\}$, where the x_i are generators of a particular polynomial ring over the field $\mathbb{Z}_p$. He comments that the same method of construction can be used to give a non-nilpotent finitely generated Engel group, thus proving that not all Engel groups are locally nilpotent.

To add to the information about Engel $\mathfrak{X}$-groups for various group-theoretic properties $\mathfrak{X}$, in 1960 Garaščuk and Supranenko [**31**] showed that a linear group over a field of characteristic zero is Engel if and only if it is locally nilpotent.

In 1998, Burns and Medvedev [**32**] showed that the commutator subgroup of a finitely generated Engel group is also finitely generated.

In 2003 Medvedev [**33**] proved that a compact Engel group is locally nilpotent.

4. Recent History of n-Engel Groups

In this section we examine more recent historical developments in the study of n-Engel groups, organized by property. The organization of this section is topical rather than chronological.

4.1. General n-Engel Groups. In 1991 Wilson [**34**] showed that a finitely generated residually finite n-Engel group is nilpotent.

In 2002, H. Smith [**35**] proved that every n-Engel group in which every subgroup is subnormal is nilpotent. In the same year, Abdollahi and Traustason

[**36**] showed that for every positive integer n, there is a positive integer $s(n)$ such that every powerful n-Engel p-group is nilpotent of class at most $s(n)$.

Traustason and Crosby will soon publish a paper [**37**] in which they show that there exist positive integers $m(n)$ and $r(n)$ such that the law $[x^{r(n)}, x_1, \ldots, x_{m(n)}] = 1$ holds for all x and for all $x_i, i = 1, \ldots, m(n)$ in any locally nilpotent n-Engel group.

In 1961, Gruenberg [**38**] showed that every n-Engel group that is solvable of derived length d and that has no elements of prime order less than n is nilpotent of class at most $(1 + 2^{n-2})^d / 2^{n-2}$.

4.2. One-Engel Groups. A group that is 1-Engel is abelian. Abelian groups have their own theory that is not in the scope of this dissertation.

4.3. Two-Engel Groups. A 2-Engel group is, in some sense, almost abelian. In 1961, W.P. Kappe [**1**] showed that condition $\mathfrak{E}_2$ implies condition $\mathfrak{B}_1$, which means that conditions $\mathfrak{B}_1$, $\mathfrak{L}(\mathfrak{N}_1)$, $\mathfrak{L}(\mathfrak{E}_1)$, and $\mathfrak{E}_2$ are equivalent. In particular, if a is an element of a 2-Engel group G, then a^G, the normal closure of a in G, is abelian. Another well-known result (see Theorem 1.6) is that any 2-Engel group is nilpotent of class at most 3, and is nilpotent of class at most 2 if there are no elements of exponent 3 in the group by part (2) of Theorem 1.6. Other important papers that discuss 2-Engel groups are by Burnside [**8**], Fite [**9**] [**10**], and Levi [**25**], and are discussed in Section 3 above.

4.4. Three-Engel Groups. Heineken [**39**] proved that all 3-Engel groups are locally nilpotent, and that if a 3-Engel group G has no elements of order 2 or 5, then G is nilpotent of class at most 4. If G has an element of order 2 but not of order 5, then G is solvable (N. Gupta, [**40**]) but not necessarily nilpotent (Baer, [**22**]). Bachmuth and Mochizuki [**41**] in 1971 gave an example of a 3-Engel group of exponent 5 that is not solvable, let alone nilpotent.

Heineken [**42**] in 1971 described a subclass of 3-Engel groups that have properties like those described by Levi and others for 2-Engel groups. Heineken's paper considers those 3-Engel groups whose cyclic subgroups are subnormal of defect 2, and shows that these groups are nilpotent of class at most 5, that weight 4 commutators in these groups have exponent 3, and that $[a, [b, [b, c]]] = 1$ for all elements a, b, and c of such groups. All 2-Engel groups satisfy all three of these conditions.

In [**2**] W.P. Kappe and L.-C. Kappe showed that $\mathfrak{E}_3$ implies $\mathfrak{L}(\mathfrak{N}_2)$, so conditions $\mathfrak{L}(\mathfrak{N}_2)$, $\mathfrak{L}(\mathfrak{E}_2)$, and $\mathfrak{E}_3$ are equivalent. Hence G is a 3-Engel group if and only if for every element a of G, the nilpotence class of a^G is 2 or smaller. In 1989 N. Gupta and Newman [**43**] showed that a 3-Engel n-generator group G is nilpotent of class at most $2n-1$ if $n > 2$, that a 3-Engel group with no elements of order 5 has nilpotence class at most $n+2$, and that the exponent of the 5th term of the lower central series of a 3-Engel group divides 20. Heineken in 1961 [**39**] showed that a 3-Engel group that has no elements of order 2 or 5 is nilpotent of class at most 4.

4.5. Four-Engel Groups. There are many results about 4-Engel groups, and for this reason the results in this subsection are given ordered by importance for the content of this dissertation rather than chronologically.

In 2005, Traustason [**44**] gave some properties of 2-generator 4-Engel groups. Later in 2005, Havas and Vaughan-Lee [**45**] proved that all 4-Engel groups are locally nilpotent, although their proof was highly dependent on computer computations in a specific group. In that same year, Traustason [**46**] published results that replace Havas and Vaughan-Lee's computer calculations with hand calculations, effectively making the proof of Havas and Vaughan-Lee computer free.

In 1980, Gupta and Levin [**3**] constructed a 4-Engel 5-group that has an element whose normal closure is not nilpotent of class 3, thus showing that $\mathfrak{E}_4$ does not imply $\mathfrak{L}(\mathfrak{N}_3)$. We examine this example in detail in Chapter 3. In 1999, Nickel [**5**] constructed a 2-group example similar to the example of Gupta and Levin for 5-groups. In 2003, Traustason [**47**] proved that the normal closure of every element of a 4-Engel group without elements of order 2 or 5 is nilpotent of class 3. In 2007, Vaughan-Lee [**4**] proved that condition $\mathfrak{E}_4$ implies condition $\mathfrak{L}(\mathfrak{E}_3)$.

Traustason [**48**] showed in 1994 that if a 4-Engel group has no elements of order 2, 3, or 5, then it is nilpotent of class at most 7. In 2002, Abdollahi and Traustason [**36**] showed that a 4-Engel group without elements of order 2 or 5 is solvable, but a 4-Engel group with elements of order 5 is not always solvable because Bachmuth and Mochizuki [**41**] construct a 3-Engel group with elements of order 5 that is not solvable. Abdollahi and Traustason [**36**] point out that a 4-Engel 2-group is not necessarily solvable because every group of exponent 4 is central-by-4-Engel (Bayes, Kautsky, and Wamsley [**49**]) and Razmyslov [**50**] (translated into English as [**51**]) constructs a non-solvable group of exponent 4 (note that if a group is not solvable, then the quotient of that group by its center remains non-solvable).

Longobardi and Maj [**52**] proved in 1997 that every right orderable 4-Engel group is nilpotent.

4.6. n-Engel Groups for Larger Values of n. It is not known in the literature if $\mathfrak{E}_5$ implies $\mathfrak{L}(\mathfrak{N}_4)$. This question will be resolved in Section 5 of Chapter 3, where we show that there are 2-groups, 3-groups, 5-groups, and 7-groups that satisfy Condition $\mathfrak{E}_5$ but not Condition $\mathfrak{L}(\mathfrak{N}_4)$. In [**4**], Vaughan-Lee constructed a 5-Engel group that has an element whose normal closure is not 4-Engel, proving that condition $\mathfrak{E}_5$ does not imply condition $\mathfrak{L}(\mathfrak{E}_4)$. Vaughan-Lee's example is a 3-group. It is not known if for other primes p there exist p-groups that satisfy condition $\mathfrak{E}_5$ but not condition $\mathfrak{L}(\mathfrak{E}_4)$. It is also not known if 5-Engel groups are locally nilpotent.

For values of n greater than 5, groups that are n-Engel have not been extensively studied. In Section 5 of Chapter 3, we show that there are groups that are 6-Engel but contain an element whose normal closure is not nilpotent of class 5. Rips and Shalev [**6**] show that for every odd prime p, there is some value of n such that for p-groups, $\mathfrak{E}_n$ does not imply $\mathfrak{L}(\mathfrak{E}_{n-1})$. The result of Rips and Shalev will be discussed in detail in Chapter 5. Gupta and Levin [**3**] also show that for every odd prime p, there is a p-group that is $(p+2)$-Engel but that does not satisfy condition $\mathfrak{L}(\mathfrak{N})$. These examples of Gupta and Levin are explained in detail in Chapter 4. It is not known if n-Engel groups are locally nilpotent for values of n larger than 5.

5. Further Questions

In Chapter 4, we show that not every n-Engel group has property $\mathfrak{L}(\mathfrak{N})$. It would be interesting to know:

QUESTION 1. Does every finitely generated n-Engel group satisfy condition $\mathfrak{L}(\mathfrak{N})$?

QUESTION 2. Does every torsion-free n-Engel group satisfy condition $\mathfrak{L}(\mathfrak{N})$?

In Chapter 5, for each prime p we find an upper bound on the least n such that a p-group with property $\mathfrak{E}_n$ need not have property $\mathfrak{L}(\mathfrak{E}_{n-1})$. It would be interesting to resolve:

PROBLEM 3. For each prime p, find an improved bound for the least n such that a p-group with property $\mathfrak{E}_n$ need not have property $\mathfrak{L}(\mathfrak{E}_{n-1})$.

The groups in Chapter 5 are not solvable, and are torsion groups. It would be of interest to answer the following quesions:

QUESTION 4. Is condition $\mathfrak{E}_n$ equivalent to condition $\mathfrak{L}(\mathfrak{E}_{n-1})$ for torsion-free groups?

QUESTION 5. Is $\mathfrak{E}_n$ equivalent to condition $\mathfrak{L}(\mathfrak{E}_{n-1})$ for solvable groups?

In Chapter 6, we show that in the variety $\mathfrak{W}_m$, Conditions $\mathfrak{E}_n$, $\mathfrak{L}(\mathfrak{E}_{n-1})$, and $\mathfrak{L}(\mathfrak{N}_{n-1})$ are equivalent when $n \geq m-2$. This suggests the following questions:

QUESTION 6. Is Condition $\mathfrak{B}_n$ equivalent to Conditions $\mathfrak{E}_n$, $\mathfrak{L}(\mathfrak{E}_{n-1})$, and $\mathfrak{L}(\mathfrak{N}_{n-1})$ for $\mathfrak{W}_m$ when $n \geq m-2$?

QUESTION 7. If the answer to Question 6 is no, is there another variety in which Condition $\mathfrak{B}_n$ is equivalent to Conditions $\mathfrak{E}_n$, $\mathfrak{L}(\mathfrak{E}_{n-1})$, and $\mathfrak{L}(\mathfrak{N}_{n-1})$ for $\mathfrak{W}_m$ when $n \geq m - 2$?

Finally, there still remains open the question:

QUESTION 8. Are n-Engel groups locally nilpotent for values of n larger than 4?

Chapter 2

Computer Background and Methodology

In this chapter we introduce some background that is needed for computer calculations in this dissertation.

1. Power Commutator Presentations

In this section, we define a presentation for a group of prime power order called a power-commutator presentation. This presentation is useful for computer computations, because, as we will see, it provides a normal form for elements of the group.

DEFINITION 2.1. Let G be a finite p-group, where p is a prime number. A *power-commutator presentation* (usually shortened to *pc presentation*) of G is a presentation of the form
$\langle a_1, \ldots, a_n \mid a_i^p = v_{ii}, 1 \leq i \leq n, [a_k, a_j] = v_{jk}, 1 \leq j < k \leq n \rangle$, where v_{jk} is a word in the elements $a_{k+1}, \ldots, a_n$. The generators $a_1, \ldots, a_n$ of a pc presentation for G are referred to as *pc generators* of G. A pc presentation for a group G of order p^m is called *consistent* if the presentation has m pc generators.

If the finite group G has exponent p, then a pc presentation for G has the form $\langle a_1, \ldots, a_n \mid a_i^p = 1, 1 \leq i \leq n, [a_k, a_j] = v_{jk}, 1 \leq j < k \leq n \rangle$, where v_{jk} is a word in the elements $a_{k+1}, \ldots, a_n$.

If the p-group G has a consistent pc presentation, then every group element can be written uniquely as a word in the normal form $a_1^{e_1} a_2^{e_2} \ldots a_n^{e_n}$, where $0 \leq e_i < p$ for $1 \leq i \leq n$ ([**53**]). Also, every group of order p^m, where p is a prime number, has a consistent pc presentation([**54**]).

Definition 2.2. Let G be a group and p a prime number. The *p-central series* $P_0(G) = G \supseteq P_1(G) \supseteq \cdots \supseteq P_{c+1}(G) = 1$ of G is defined by $P_0(G) = G$ and $P_i(G) = [G, P_{i-1}(G)](P_{i-1}(G))^p$ for $i \geq 1$. Because G is a finite p-group, the p-central series of G eventually ends in the trivial group. The *p-class of G* is the smallest integer c such that $P_{c+1}(G) = 1$.

We note that if G is a group of exponent p, then the p-central series of G is the same as the lower central series of G.

Definition 2.3. Let G be a finite p-group and let $\{a_1, \ldots, a_n\}$ be a set of pc generators for a consistent pc presentation of G. Suppose the composition series $\langle 1 \rangle < \langle a_n \rangle < \langle a_n, a_{n-1} \rangle < \cdots < G$ refines the p-central series for G. We define a *weight function* w on the pc generators $\{a_1, \ldots, a_n\}$ by $w(a_i) = j$ if a_i is contained in $P_{j-1}(G)$ but not in $P_j(G)$. In this case the pc presentation is called a *weighted pc presentation*.

In the following, we will at times use the term *p-weight* for the weight function on a pc-presentation.

Sims [**55**, p 447] shows that every finite p-group has a weighted pc presentation. He also gives the following properties of a weighted pc presentation:

(1) $w(a_i) \leq w(a_{i+1})$ for $1 \leq i < n$.
(2) For $1 \leq i \leq n$ there is a power relation $a_i^p = v_{ii}$, where the pc generators that occur in v_{ii} all have weight at least $w(a_i) + 1$.
(3) For the relation $[a_i, a_j] = v_{ji}$, the word v_{ji} contains only generators of weight at least $w(a_i) + w(a_j)$.
(4) If a_k is a pc generator with $w(a_k) > 1$, then one of the following holds:
    (a) There are indices i and j such that $a_k = [a_i, a_j]$ with $w(a_i) = w(a_k) - 1$ and $w(a_j) = 1$.
    (b) There exists an index i such that $a_k = a_i^p$ with $w(a_i) = w(a_k) - 1$.

If G has exponent p, then these properties show that any generator a_k of weight greater than one is the commutator of a generator a_j of weight $w(a_k) - 1$ and a generator a_i of weight one. Also, the weight of a pc generator is the same as its commutator weight.

We also see that a weighted pc presentation must have at least one generator of each weight up to and including the p-class.

DEFINITION 2.4. Let G be a finite p-group and let $\{a_1, \ldots, a_n\}$ be a set of pc generators for a weighted pc presentation of G. Let d be the p-class of G. In this dissertation the *pc structure* of G is a finite sequence of integers $b_1, \ldots, b_d$, where b_i is the number of pc generators of G that are of weight i or less.

2. Computer Background

The computer package GAP (short for Groups, Algorithms, and Programming) [**56**] is used for many calculations in Chapter 3. In particular, the package ANUPQ [**57**] is used for computing quotients. In the construction of various examples the function Pq from the package ANUPQ is frequently used. To use this function to explore a group G, we input a free group on a certain number of generators. We also give a list of relators that are satisfied in G and a prime p. We optionally can provide a bound b that specifies the last term of the p-central series to calculate. The function Pq then determines a set of factor groups of the subgroups of the p-central series for G. It stops calculating the factor groups at p-class b, or else stops when the p-central series becomes trivial, whichever comes first. We may also impose exponent laws on G via an option we give the Pq function, and impose other laws (usually the n-Engel law for a small integer n) if needed. As GAP and Pq are calculating, GAP gives a list of the sum of the number of pc generators of the factor groups calculated so far. In fact, we read the pc structure of G directly from these numbers.

Many of the groups we investigate have exponent p, in which case the p-central series is the same as the lower central series. In this case the p-weights of the pc generators are their standard commutator weights. Also, the p-class of a group of exponent p is the same as the nilpotence class of the group.

In Chapter 5, we also use the computer software Maple [**58**] for number theory calculations.

3. Building a Presentation

For this section, fix a prime number p. We show how to construct a pc presentation for a finite p-group G using GAP. Choose a number m of group generators for G. Then the number of pc generators of G of weight 1 is also m. We also choose a p-class c. Usually we also pick an exponent e, which is a power of p. We now pick an ordering for the weight 1 generators and label them a_1 to a_m. If we are building the group in GAP, then we fulfill this step by starting out with a free group on m generators. For the rest of this section, we will construct examples using $m = 2$, $p = 5$, $e = 5$, and $c = 3$. The following GAP code will give G :

```
RequirePackage("anupq");
f:=FreeGroup(2);
g:=Pq(f: Prime:=5, Exponent:=5, ClassBound:=3);
```

Now we decide which weight 2 commutators we want as pc generators in G, and what power generators we want. The power generators will all be p powers of weight 1 generators. Because G has exponent p, we need only pick weight 2 commutators since a_i^p is always 1. In GAP, it is easier to pick which of the possible generators we wish to be trivial. This is done by adding weight 2 commutators to the list of relators which we give GAP. For example,

```
RequirePackage("anupq");
f:=FreeGroup(2);
rels:=["[f1,f2]"];
g:=Pq(f: Prime:=5, Exponent:=5, Relators:=rels, ClassBound:=3);
```

This calculation gives the group $C_5 \times C_5$. We have made the only possible weight 2 pc generator $[a_1, a_2]$ trivial, so our pc presentation has 5-class 1. For this reason, in the following we will assume that G has no relators of weight 2 so that we will obtain a weight 2 pc generator. If the free group had more than 2 generators, we could have a weight 2 relator and still have other weight 2 pc generators.

We continue by working with the possible weight 3 pc generators. In the previous step, we decided that there is one weight 2 pc generator, $[a_1, a_2]$. The possible weight 3 pc generators are $[a_1, a_2, a_1]$ and $[a_1, a_2, a_2]$. If we had not specified that we were working with a group of exponent 5, then we would also need to consider the 5th power of each weight 2 generator, such as $[a_1, a_2]^5$. For our example, we decide that $[a_1, a_2, a_1]$ is a pc generator and that $[a_1, a_2, a_2]$ is trivial. When we are using GAP, it is again easier to indicate which possible weight 3 simple commutators in the group generators we wish to be trivial. Note that not all weight 3 simple commutators are necessarily either pc generators or relators. We can construct the group G in GAP by the code:

```
RequirePackage("anupq");
f:=FreeGroup(2);
rels:=["[f1,f2,f2]"];
g:=Pq(f: Prime:=5, Exponent:=5, Relators:=rels, ClassBound:=3);
```

We continue this process until we reach the p-class that we decided on at first. In our example, we have reached our 5-class of 3, so we stop.

In practice, we usually want to create a group that has additional properties. It is common to want to add a law to a group. For example, the following GAP code includes the 3-Engel law.

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=["[f1,f2,f2]"];
g:=Pq(f: Prime:=5, Exponent:=5, ClassBound:=5, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b]);
end]);
```

Sometimes we want to construct a group that does not have a particular property P. If, for instance, we want a group where the nilpotence class of the normal closure of the first generator is greater than 2, then we add relators as shown above, testing the nilpotence class of the normal closure of the first generator after each relator is added. If the group which has the questioned relator has property P, then we know we do not want that relator. This is the method used to generate many of the examples in Chapter 3.

Chapter 3

Examples Where $\mathfrak{E}_n$ Does Not Imply $\mathfrak{L}(\mathfrak{N}_{n-1})$

1. Gupta and Levin Counterexample

In this section we investigate the details of an example by Gupta and Levin [**3**] that shows that condition $\mathfrak{L}(\mathfrak{N}_3)$ is not implied by condition $\mathfrak{E}_4$.

1.1. Example of Gupta and Levin. In this subsection, we construct in detail the first published example, by Gupta and Levin, of a 4-Engel group that has an element whose normal closure in the group is not nilpotent of class 3. Our explanation here contains many more details than the original explanation in [**3**].

We will need the following lemmas:

Lemma 3.1 (Chapter 5 of [**59**]). *Let A be an associative algebra and L its related Lie algebra. Let G be a group whose generators are of the form $1+a$, where a is in L. Then $[1+a_1, 1+a_2, \cdots, 1+a_n] = 1 + (a_1, a_2, \cdots, a_n) + t$, where t is a sum of Lie commutators of weight greater than n in L.*

Lemma 3.2 (Lemma 4.1 in [**3**]). *Let p be an odd prime. Let G be a group of exponent p that is nilpotent of class at most $p+1$. Then G is $(p-1)$-Engel if and only if G satisfies the law $[x, y, x, {}_{p-2}y] = 1$.*

In Lemma 3.4, we will use the Diamond Lemma for algebras [**60**]. We follow the exposition of Appendix I.11, pages 97-102, from [**61**].

Definition 3.1. Let F be a free algebra on a set of generators $\langle X\rangle$ and A an algebra that is a quotient of F. Then we can write $A = F/\langle w_\sigma - f_\sigma \mid \sigma \in \Sigma\rangle$, where $f_\sigma \in F$ and w_σ is a word in X. We call $S = \{(w_\sigma, f_\sigma) \mid \sigma \in \Sigma\}$ a *reduction system*, and each (w_σ, f_σ) in S we call a *reduction.* An element of F is called *irreducible* if no reduction from S alters it. An *ambiguity* is a word in X to which we could apply more than one reduction. A reduction system S and an ordering $\leq$ on the set of words on X are *compatible* when for every $\sigma \in \Sigma$, the element f_σ is a linear combination of words $w < w_\sigma$.

LEMMA 3.3 (The Diamond Lemma). *(see* [**60**]*) Let $F = k\langle X\rangle$ be a free algebra on a set X and W the free monoid on X. Let $S = \{(w_\sigma, f_\sigma) \mid \sigma \in \Sigma\}$ be a reduction system of F, and $\leq$ a semigroup ordering on W which is compatible with S and satisfies the Descending Chain Condition. If all overlap and inclusion ambiguities are resolvable, then the cosets $\bar{w}$, for irredicible words $w \in W$, form a basis for the factor algebra $F/\langle w_\sigma - f_\sigma \mid \sigma \in \Sigma\rangle$.*

Let R be the free associative algebra in three non-commuting indeterminates, x_0, x_1 and x_2, over the field $\mathbb{Z}_5$. Construct an ideal in the following manner:

Set $S_1 =$\{monomials of degree 7 in R\}, $S_2 =$\{monomials in R that are of degree 2 in x_1 or of degree 2 in x_2\}, $S_3 = \{x_1x_2,\ x_2x_1,\ x_0^3,\ x_0^2x_1,\ x_0^2x_2\}$, and $S_4 = \{x_1x_0x_2 + x_2x_0x_1\}$. Let I be the ideal in R generated by S_1, S_2, S_3, and S_4. Set $A = R/I$.

LEMMA 3.4. *(Gupta and Levin* [**3**]*) Let H be the set of all products of powers of the elements $1 + x_0$, $1 + x_1$, and $1 + x_2$ in A (including the empty product, which is 1 in A). Then H is a multiplicative group that has exponent 5, has order at most 5^{20}, is nilpotent of class exactly 6, is 4-Engel, and contains an element whose normal closure in H is not nilpotent of class 3.*

PROOF. We begin with an examination of the monomials in A using the Diamond Lemma (Lemma 3.3). We can see that A is determined by the following reduction system S: for each element w of S_1, S_2, and S_3 we have a reduction $(w, 0)$. The only other reduction needed is $(x_2x_0x_1, -x_1x_0x_2)$. In order to apply the Diamond Lemma, we must define a partial ordering on the free monoid $\langle x_0, x_1, x_2\rangle$ that has the descending chain condition and is compatible with S. We consider the length-lexicographic order, first defining $x_0 < x_1 < x_2$. In this ordering, for words w and v, we define $w < v$ if the length of w is less than the length of v, or else if w and v have the same length and w precedes v in the lexicographic order. This ordering is consistent with every $(w, 0)$ reduction because $0 < w$. It also is consistent with $(x_2x_0x_1, -x_1x_0x_2)$ because $x_1x_0x_2 < x_2x_0x_1$. This ordering also satisfies the descending chain condition. Thus, we have an ordering required by the Diamond Lemma. Now, we need to show that all ambiguities in S are resolvable. We do not need to be concerned about reductions $(w, 0)$ and $(v, 0)$, because the use of either reduction first will still result in 0. Thus, all we need to consider is an ambiguity between reductions $(w, 0)$ and $(x_2x_0x_1, -x_1x_0x_2)$. If $w \in S_1$, there is no problem because the reduction $(x_2x_0x_1, -x_1x_0x_2)$ does not alter the length of words. Similarly, $(x_2x_0x_1, -x_1x_0x_2)$ does not alter the weight of a word in x_1 or x_2, so ambiguities with $w \in S_2$ are also resolvable. The only remaining case is when $w \in S_3$. We see that w cannot be x_0^3, x_2x_1, or $x_0^2x_1$ because there is no possible overlap or containment with $x_2x_0x_1$. If $w = x_1x_2$, the only possible minimal overlaps are the monomials $x_1x_2x_0x_1$ and $x_2x_0x_1x_2$. If we first apply $(x_2x_0x_1, -x_1x_0x_2)$, we get $-x_1^2x_0x_2$ and $-x_1x_0x_2^2$ and we then apply a reduction from S_2 and get 0. If we apply the reduction from S_2 first, the result in either case is 0, so the ambiguity is resolvable. If $w = x_0^2x_2$, the only possible minimal overlap is the monomial $x_0^2x_2x_0x_1$. If we first apply $(x_2x_0x_1, -x_1x_0x_2)$, then we get $-x_0^2x_1x_0x_2$ and apply the reduction $(x_0^2x_1, 0)$ to get 0. If we apply the reduction $(x_0^2x_1, 0)$ first, we again get 0. Thus, by the Diamond Lemma, the irreducible words form a basis for A. In Table 1 we list all nontrivial irreducible words, namely, the words to which we can not apply any reduction. Thus, the monomials in Table 1, together with 1, provide a basis for A.

We treat A as consisting of all linear combinations over $\mathbb{Z}_5$ of monomials from Table 1. Then H contains only elements of the form $1 + y$, where y is a linear combination of monomials from Table 1.

Table 1: Irreducible Monomials in A

x_0	x_1	x_2		
x_0x_1	x_0x_2	x_0^2	x_1x_0	x_2x_0
$x_0x_1x_0$	$x_0x_2x_0$	$x_1x_0x_2$	$x_1x_0^2$	$x_2x_0^2$
$x_0x_1x_0x_2$	$x_1x_0x_2x_0$	$x_0x_1x_0^2$	$x_0x_2x_0^2$	
$x_0x_1x_0x_2x_0$	$x_1x_0x_2x_0^2$			
$x_0x_1x_0x_2x_0^2$				

We verify the following:

(1) The set H is closed under multiplication

(2) H contains an identity (which also shows that H is nonempty)

(3) Each element in H has an inverse in H

(4) H is of exponent 5

(5) H is of order $\leq 5^{20}$

(6) H is a 4-Engel group

(7) H is nilpotent of class 6

(8) There is at least one element of H whose normal closure in H is not nilpotent of class 3.

(1) and (2) follow immediately from the definition of H.

We now show that all elements of H are of exponent 5. Let m be an element of H. Then $m = 1 + n$, where n is a linear combination of monomials from Table 1. The coefficients in m are in $\mathbb{Z}_5$, so $m^5 = 1 + n^5$. However, every term of n^5 is a product of 5 monomials, and monomials of degree greater than 6 are trivial in A, so we may assume that n contains only monomials of degree 1 or 2. Let $n = a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_0 + a_7x_2x_0$. In the following calculations, we gather together and label with uppercase letters terms of weight large enough to become trivial by the end of the calculation.

$$\begin{aligned} n^2 &= (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_0 + a_7x_2x_0)^2 \\ &= a_0^2x_0^2 + a_0a_1x_0x_1 + a_0a_2x_0x_2 + a_0a_1x_1x_0 + a_0a_2x_2x_0 + (a_0a_6 + a_4a_0)x_0x_1x_0 \\ &\quad + (a_0a_7 + a_5a_0)x_0x_2x_0 + (a_1a_3 + a_6a_0)x_1x_0^2 + (a_2a_3 + a_7a_0)x_2x_0^2 \\ &\quad + (a_1a_5 - a_2a_4 + a_6a_2 - a_7a_1)x_1x_0x_2 + N, \end{aligned} \tag{4}$$

where N is a sum of monomials of degree 4 or more. Further,

$$\begin{aligned} n^4 = (&a_0^2x_0^2 + a_0a_1x_0x_1 + a_0a_2x_0x_2 + a_0a_1x_1x_0 + a_0a_2x_2x_0 + (a_0a_6 + a_4a_0)x_0x_1x_0 \\ &+ (a_0a_7 + a_5a_0)x_0x_2x_0 + (a_1a_3 + a_6a_0)x_1x_0^2 + (a_2a_3 + a_7a_0)x_2x_0^2 \\ &+ (a_1a_5 - a_2a_4 + a_6a_2 - a_7a_1)x_1x_0x_2 + N)^2 \\ = &a_0^3a_1x_0x_1x_0^2 + a_0^3a_2x_0x_2x_0^2 + (a_0^2a_1a_5 - a_0^2a_2a_4)x_1x_0x_2x_0^2 + M, \end{aligned} \tag{5}$$

where M is a sum of monomials of degree 6 or more. Finally,

$$\begin{aligned} n^5 &= (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_0 + a_7x_2x_0) \\ &\quad \cdot (a_0^3a_1x_0x_1x_0^2 + a_0^3a_2x_0x_2x_0^2 + (a_0^2a_1a_5 - a_0^2a_2a_4)x_1x_0x_2x_0^2 + M) \\ &= 0. \end{aligned} \tag{6}$$

We conclude that H is of exponent 5, which shows (4).

(3) follows since every nonidentity element of H is of order 5.

(5) holds because there are 5^{20} linear combinations over $\mathbb{Z}_5$ of the monomials in Table 1.

Next we prove (7). Let a, b, c, d, e, f, and g be elements of A, so that $1+a$, $1+b$, $1+c$, $1+d$, $1+e$, $1+f$, and $1+g$ are elements of H. By Lemma 3.1, $[1+a, 1+b, 1+c, 1+d, 1+e, 1+f, 1+g] = 1 + (a, b, c, d, e, f, g) + t$, where t is a sum of Lie commutators of weight 8 or more. However, in A only terms of weight 6 or less are nontrivial, so t is trivial. Similarly, (a, b, c, d, e, f, g) has weight 7 and so is trivial. Thus a commutator of weight 7 or more in H is trivial. Hence the nilpotence class of H is less than or equal to 6. To show that H has nilpotence class 6, we consider the commutator $[1+a, 1+b, 1+b, 1+b, 1+c, 1+b]$. By Lemma 3.1, $[1+a, 1+b, 1+b, 1+b, 1+c, 1+b] = 1 + (a, b, b, b, c, b)$. Now we let $a = x_1$, $b = x_0$, and $c = x_2$. Then

$$
\begin{aligned}
&[1+x_1, 1+x_0, 1+x_0, 1+x_0, 1+x_2, 1+x_0] \\
&= 1 + (x_1, x_0, x_0, x_0, x_2, x_0) \\
&= 1 + (x_1x_0 - x_0x_1, x_0, x_0, x_2, x_0) \\
&= 1 + (x_1x_0x_0 - x_0x_1x_0 - x_0x_1x_0 + x_0x_0x_1, x_0, x_2, x_0) \\
&= 1 + (x_1x_0x_0 - 2x_0x_1x_0 + x_0x_0x_1, x_0, x_2, x_0) \\
&= 1 + (x_1x_0x_0x_0 - 2x_0x_1x_0x_0 + x_0x_0x_1x_0 - x_0x_1x_0x_0 \\
&\qquad + 2x_0x_0x_1x_0 - x_0x_0x_0x_1, x_2, x_0) \\
&= 1 + (-3x_0x_1x_0^2, x_2, x_0) \\
&= 1 + (3x_2x_0x_1x_0^2, x_0) \\
&= 1 - 3x_0x_2x_0x_1x_0^2.
\end{aligned}
\tag{7}
$$

Because $[1+x_1, 1+x_0, 1+x_0, 1+x_0, 1+x_2, 1+x_0] \in H$ is not trivial, the nilpotence class of H is exactly 6.

Now we can prove (6). To show that H is 4-Engel by Lemma 3.2, we show that $[x, y, x, y, y, y] = 1$ for all $x, y \in H$. Let $x = 1 + j$ and $y = 1 + k$ where i and j are in A. By Lemma 3.1, it suffices to show that $(j, k, j, k, k, k) = 0$.

We may assume without loss of generality that j and k contain only monomials of degree 1, because any greater degree terms in (j, k, j, k, k, k) would give terms that are in S_1 and so are trivial in H. Thus $j = j_0x_0 + j_1x_1 + j_2x_2$ and $k = k_0x_0 + k_1x_1 + k_2x_2$, where j_0, j_1, j_2, k_0, k_1, and k_2 are in $\mathbb{Z}_5$.

Now,

$$
\begin{aligned}
jk &= (j_0x_0 + j_1x_1 + j_2x_2)(k_0x_0 + k_1x_1 + k_2x_2) \\
&= j_0k_0x_0^2 + j_0k_1x_0x_1 + j_0k_2x_0x_2 \\
&\quad + j_1k_0x_1x_0 + j_1k_1x_1^2 + j_1k_2x_1x_2 \\
&\quad + j_2k_0x_2x_0 + j_2k_1x_2x_1 + j_2k_2x_2^2.
\end{aligned}
\tag{8}
$$

Because x_1^2, x_2^2, x_1x_2, and x_2x_1 are all in I, we have

$$jk = j_0k_0x_0^2 + j_0k_1x_0x_1 + j_0k_2x_0x_2 + j_1k_0x_1x_0 + j_2k_0x_2x_0.$$

By symmetry,

$$kj = j_0k_0x_0^2 + j_1k_0x_0x_1 + j_2k_0x_0x_2 + j_0k_1x_1x_0 + j_0k_2x_2x_0.$$

Thus

$$\begin{aligned}
(j,k) &= jk - kj \\
&= (j_0k_0x_0^2 + j_0k_1x_0x_1 + j_0k_2x_0x_2 + j_1k_0x_1x_0 + j_2k_0x_2x_0) \\
&\quad - (j_0k_0x_0^2 + j_1k_0x_0x_1 + j_2k_0x_0x_2 + j_0k_1x_1x_0 + j_0k_2x_2x_0) \\
&= (j_0k_1 - j_1k_0)x_0x_1 + (j_0k_2 - j_2k_0)x_0x_2 \\
&\quad - (j_0k_1 - j_1k_0)x_1x_0 - (j_0k_2 - j_2k_0)x_2x_0.
\end{aligned} \tag{9}$$

In the following, we use the notations ℓ_1 for $j_0k_1 - j_1k_0$ and ℓ_2 for $j_0k_2 - j_2k_0$. Then

$$(j,k) = \ell_1x_0x_1 + \ell_2x_0x_2 - \ell_1x_1x_0 - \ell_2x_2x_0.$$

We also abbreviate $j_1k_2 - j_2k_1$ by ℓ_3. Then $\ell_2k_1 - \ell_1k_2 = \ell_3k_0$. Continuing,

$$\begin{aligned}
(j,k,j) &= (j,k)j - j(j,k) \\
&= (\ell_1x_0x_1 + \ell_2x_0x_2 - \ell_1x_1x_0 - \ell_2x_2x_0)(j_0x_0 + j_1x_1 + j_2x_2) \\
&\quad - (j_0x_0 + j_1x_1 + j_2x_2)(\ell_1x_0x_1 + \ell_2x_0x_2 - \ell_1x_1x_0 - \ell_2x_2x_0) \\
&= \ell_1j_0x_0x_1x_0 + \ell_2j_0x_0x_2x_0 - \ell_1j_0x_1x_0^2 \\
&\quad - \ell_1j_2x_1x_0x_2 - \ell_2j_0x_2x_0^2 - \ell_2j_1x_2x_0x_1 \\
&\quad + j_0\ell_1x_0x_1x_0 + j_0\ell_2x_0x_2x_0 - j_1\ell_2x_1x_0x_2 - j_2\ell_1x_2x_0x_1 \\
&= 2\ell_1j_0x_0x_1x_0 + 2\ell_2j_0x_0x_2x_0 - \ell_1j_0x_1x_0^2 - \ell_2j_0x_2x_0^2 \\
&= j_0\left(2\ell_1x_0x_1x_0 + 2\ell_2x_0x_2x_0 - \ell_1x_1x_0^2 - \ell_2x_2x_0^2\right).
\end{aligned} \tag{10}$$

Further,

$$
\begin{aligned}
(j,k,j,k) &= (j,k,j)k - k(j,k,j) \\
&= j_0\big((2\ell_1x_0x_1x_0 + 2\ell_2x_0x_2x_0 - \ell_1x_1x_0^2 - \ell_2x_2x_0^2)(k_0x_0 + k_1x_1 + k_2x_2) \\
&\quad - (k_0x_0 + k_1x_1 + k_2x_2)(2\ell_1x_0x_1x_0 + 2\ell_2x_0x_2x_0 - \ell_1x_1x_0^2 - \ell_2x_2x_0^2)\big) \\
&= j_0\big(2\ell_1k_0x_0x_1x_0^2 + 2\ell_2k_0x_0x_2x_0^2 + 2\ell_2k_1x_0x_2x_0x_1 + 2\ell_1k_2x_0x_1x_0x_2 \\
&\quad + k_0\ell_1x_0x_1x_0^2 + k_0\ell_2x_0x_2x_0^2 - 2k_1\ell_2x_1x_0x_2x_0 - 2k_2\ell_1x_2x_0x_1x_0\big) \\
&= j_0\big(3\ell_1k_0x_0x_1x_0^2 + 3\ell_2k_0x_0x_2x_0^2 \\
&\quad - 2(\ell_2k_1 - \ell_1k_2)x_0x_1x_0x_2 - 2(\ell_2k_1 - \ell_1k_2)x_1x_0x_2x_0\big) \\
&= j_0\big(3\ell_1k_0x_0x_1x_0^2 + 3\ell_2k_0x_0x_2x_0^2 - 2\ell_3k_0x_0x_1x_0x_2 - 2\ell_3k_0x_1x_0x_2x_0\big] \\
&= 3j_0k_0\big(\ell_1x_0x_1x_0^2 + \ell_2x_0x_2x_0^2 + \ell_3x_0x_1x_0x_2 + \ell_3x_1x_0x_2x_0\big).
\end{aligned}
\tag{11}
$$

Thus,

$$
\begin{aligned}
(j,k,j,k,k) &= (j,k,j,k)k - k(j,k,j,k) \\
&= 3j_0k_0\big((\ell_1x_0x_1x_0^2 + \ell_2x_0x_2x_0^2 + \ell_3x_0x_1x_0x_2 \\
&\quad + \ell_3x_1x_0x_2x_0)(k_0x_0 + k_1x_1 + k_2x_2) \\
&\quad - (k_0x_0 + k_1x_1 + k_2x_2)(\ell_1x_0x_1x_0^2 \\
&\quad + \ell_2x_0x_2x_0^2 + \ell_3x_0x_1x_0x_2 + \ell_3x_1x_0x_2x_0)\big) \\
&= 3j_0k_0\big(\ell_3k_0x_0x_1x_0x_2x_0 + \ell_3k_0x_1x_0x_2x_0^2 \\
&\quad - k_0\ell_3x_0x_1x_0x_2x_0 - k_1\ell_2x_1x_0x_2x_0^2 - k_2\ell_1x_2x_0x_1x_0^2\big) \\
&= 3j_0k_0\big(\ell_3k_0x_1x_0x_2x_0^2 - k_1\ell_2x_1x_0x_2x_0^2 - k_2\ell_1x_2x_0x_1x_0^2\big) \\
&= 3j_0k_0\big(\ell_3k_0 - k_1\ell_2 + k_2\ell_1\big)x_1x_0x_2x_0^2.
\end{aligned}
\tag{12}
$$

Because $\ell_2k_1 - \ell_1k_2 = \ell_3k_0$, it follows that $(j,k,j,k,k,k) = 0$, which completes the proof of (6). It should be noted that because of restrictions in the statement of Lemma 3.1, we have not proved that $[1+j, 1+k, 1+j, 1+k, 1+k]$ is trivial in H, only that $[1+j, 1+k, 1+j, 1+k, 1+k]$ is $1+s$, where s is a sum of weight six Lie commutators in A.

Finally to prove (8), we show that $[1+x_1, 1+x_0, 1+x_0, 1+x_0, 1+x_2, 1+x_0]$ is in $\gamma_4(\langle 1+x_0\rangle^H)$. We showed in the proof of (7) that

$[1+x_1, 1+x_0, 1+x_0, 1+x_0, 1+x_2, 1+x_0]$ is not trivial, so it will follow that $\gamma_4(\langle 1+x_0\rangle^H)$ is not trivial, which will complete the proof.

We first notice that $[1+x_1, 1+x_0] = ((1+x_0)^{-1})^{(1+x_1)}(1+x_0)$, so $[1+x_1, 1+x_0]$ is in $\langle 1+x_0\rangle^H$. Then because $(1+x_0)$ is also in $\langle 1+x_0\rangle^H$, we know that $[1+x_1, 1+x_0, 1+x_0]$ lies in $\gamma_2(\langle 1+x_0\rangle^H)$. Similarly, $[1+x_1, 1+x_0, 1+x_0, 1+x_0]$ is contained in $\gamma_3(\langle 1+x_0\rangle^H)$. Thus $[1+x_1, 1+x_0, 1+x_0, 1+x_0, 1+x_2] =$
$[1+x_1, 1+x_0, 1+x_0, 1+x_0]^{-1}([1+x_1, 1+x_0, 1+x_0, 1+x_0])^{(1+x_2)}$, so $[1+x_1, 1+x_0, 1+x_0, 1+x_0, 1+x_2]$ is an element of $\gamma_3(\langle 1+x_0\rangle^H)$ and therefore $[1+x_1, 1+x_0, 1+x_0, 1+x_0, 1+x_2, 1+x_0]$ is contained in $\gamma_4(\langle 1+x_0\rangle^H)$.

Hence the element $(1+x_0)$ of H has normal closure that is not nilpotent of class 3. Thus all the claimed properties of H are verified. $\square$

Lemma 3.4 immediately proves the major result of this section.

THEOREM 3.5. *(Gupta and Levin, [**3**]) There exists a 4-Engel 5-group H with an element a whose normal closure a^H is not nilpotent of class 3.*

1.2. Relators. In this subsection, we show that certain commutators in the Gupta and Levin example, which we call H, are trivial. First, we need a lemma that gives the inverses of some of the elements of H.

LEMMA 3.6. *In H, if w is a sum of words in x_0, x_1, and x_2, and either every word in w is of degree at least one in x_1 or every word in w is of degree at least one in x_2, then $(1+w)^{-1} = 1-w$. In particular, $(1+x_1)^{-1} = 1-x_1$ and $(1+x_2)^{-1} = 1-x_2$. Also, $(1+x_0)^{-1} = 1-x_0+x_0^2$.*

PROOF. We note that $(1+w)(1-w) = 1-w^2$. Because w^2 contains only words of degree at least 2 in either x_1 or x_2, we see w^2 is contained in S_2 and so is 0. This proves that $(1+w)(1-w) = 1$, so $(1+w)^{-1} = 1-w$. Also,
$(1+x_0)(1-x_0+x_0^2) = (1-x_0+x_0^2+x_0-x_0^2+x_0^3) = 1$, which completes the proof. $\square$

Lemma 3.7. *In H the commutators $[1+x_1, 1+x_2]$, $[1+x_0, 1+x_1, 1+x_1]$, $[1+x_0, 1+x_2, 1+x_2]$, and $[1+x_0, 1+x_1, 1+x_2, 1+x_0, 1+x_0, 1+x_0]$ are all trivial. Further, $[1+x_0, 1+x_1, 1+x_2] = [1+x_0, 1+x_2, 1+x_1]^{-1}$.*

Proof. To prove this lemma, we calculate the commutators $[1+x_1, 1+x_2]$, $[1+x_0, 1+x_1, 1+x_1]$, $[1+x_0, 1+x_2, 1+x_2]$, and $[1+x_0, 1+x_1, 1+x_2, 1+x_0, 1+x_0, 1+x_0]$. First, using Lemma 3.6,

$$
\begin{aligned}
[1+x_1, 1+x_2] &= (1+x_1)^{-1}(1+x_2)^{-1}(1+x_1)(1+x_2) \\
&= (1-x_1)(1-x_2)(1+x_1+x_2) \\
&= (1-x_1)(1+x_1+x_2-x_2) \\
&= (1-x_1)(1+x_1) \\
&= 1.
\end{aligned} \tag{13}
$$

Also, again using 3.6,

$$
\begin{aligned}
[1+x_0, 1+x_1] &= (1+x_0)^{-1}(1+x_1)^{-1}(1+x_0)(1+x_1) \\
&= (1-x_0+x_0^2)(1-x_1)(1+x_0+x_1+x_0x_1) \\
&= (1-x_0+x_0^2)(1+x_0+x_1+x_0x_1-x_1-x_1x_0) \\
&= (1-x_0+x_0^2)(1+x_0+x_0x_1-x_1x_0) \\
&= 1+x_0+x_0x_1-x_1x_0-x_0-x_0^2+x_0x_1x_0+x_0^2 \\
&= 1+x_0x_1-x_1x_0+x_0x_1x_0.
\end{aligned} \tag{14}
$$

It follows that

$$
\begin{aligned}
[1+x_0, 1+x_1, 1+x_1] &= (1+x_0x_1-x_1x_0+x_0x_1x_0)^{-1}(1+x_1)^{-1} \\
&\qquad \cdot (1+x_0x_1-x_1x_0+x_0x_1x_0)(1+x_1) \\
&= (1-x_0x_1+x_1x_0-x_0x_1x_0)(1-x_1) \\
&\qquad \cdot (1+x_0x_1-x_1x_0+x_0x_1x_0+x_1) \\
&= (1-x_0x_1+x_1x_0-x_0x_1x_0) \\
&\qquad \cdot (1+x_0x_1-x_1x_0+x_0x_1x_0+x_1-x_1) \\
&= (1-x_0x_1+x_1x_0-x_0x_1x_0) \\
&\qquad \cdot (1+x_0x_1-x_1x_0+x_0x_1x_0) \\
&= 1.
\end{aligned} \tag{15}
$$

By symmetry, $[1+x_0, 1+x_2, 1+x_2] = 1$. In addition, using (14) again, along with Lemma 3.6,

$$\begin{aligned}
[1+x_0, 1+x_1, 1+x_2] &= (1+x_0x_1 - x_1x_0 + x_0x_1x_0)^{-1}(1+x_2)^{-1} \\
&\quad \cdot (1+x_0x_1 - x_1x_0 + x_0x_1x_0)(1+x_2) \\
&= (1-x_0x_1 + x_1x_0 - x_0x_1x_0)(1-x_2)(1+x_0x_1 \\
&\quad - x_1x_0 + x_0x_1x_0 + x_2 - x_1x_0x_2 + x_0x_1x_0x_2) \\
&= (1-x_0x_1 + x_1x_0 - x_0x_1x_0)(1+x_0x_1 \\
&\quad - x_1x_0 + x_0x_1x_0 + x_2 - x_1x_0x_2 \\
&\quad + x_0x_1x_0x_2 - x_2 - x_2x_0x_1 - x_2x_0x_1x_0) \\
&= (1-x_0x_1 + x_1x_0 - x_0x_1x_0)(1+x_0x_1 - x_1x_0 \\
&\quad + x_0x_1x_0 + x_0x_1x_0x_2 - x_2x_0x_1x_0) \\
&= 1 + x_0x_1x_0x_2 - x_2x_0x_1x_0.
\end{aligned} \tag{16}$$

By symmetry, $[1+x_0, 1+x_2, 1+x_1] = 1 + x_0x_2x_0x_1 - x_1x_0x_2x_0$
$= [1+x_0, 1+x_1, 1+x_2]^{-1}$.

In the following, we use z to denote $[1+x_0, 1+x_1, 1+x_2]$. Using (16) and Lemma 3.6,

$$\begin{aligned}
[z, 1+x_0] &= (1+x_0x_1x_0x_2 - x_2x_0x_1x_0)^{-1}(1+x_0)^{-1} \\
&\quad \cdot (1+x_0x_1x_0x_2 - x_2x_0x_1x_0)(1+x_0) \\
&= (1-x_0x_1x_0x_2 + x_2x_0x_1x_0)(1-x_0+x_0^2) \\
&\quad \cdot (1+x_0x_1x_0x_2 - x_2x_0x_1x_0 + x_0 + x_0x_1x_0x_2x_0 - x_2x_0x_1x_0^2) \\
&= (1-x_0x_1x_0x_2 + x_2x_0x_1x_0)(1+x_0x_1x_0x_2 \\
&\quad - x_2x_0x_1x_0 + x_0 + x_0x_1x_0x_2x_0 - x_2x_0x_1x_0^2 \\
&\quad - x_0 + x_0x_2x_0x_1x_0 - x_0^2 + x_0x_2x_0x_1x_0^2 + x_0^2) \\
&= (1-x_0x_1x_0x_2 + x_2x_0x_1x_0)(1+x_0x_1x_0x_2 \\
&\quad - x_2x_0x_1x_0 - x_2x_0x_1x_0^2 + x_0x_2x_0x_1x_0^2) \\
&= (1 - x_2x_0x_1x_0^2 + x_0x_2x_0x_1x_0^2).
\end{aligned} \tag{17}$$

Further,

$$[z, 1+x_0, 1+x_0] = (1 - x_2x_0x_1x_0^2 + x_0x_2x_0x_1x_0^2)^{-1}(1+x_0)^{-1}$$

$$
\begin{aligned}
&\cdot (1 - x_2x_0x_1x_0^2 + x_0x_2x_0x_1x_0^2)(1 + x_0)\\
&= (1 + x_2x_0x_1x_0^2 - x_0x_2x_0x_1x_0^2)(1 - x_0 + x_0^2)\\
&\quad \cdot (1 - x_2x_0x_1x_0^2 + x_0x_2x_0x_1x_0^2 + x_0)\\
&= (1 + x_2x_0x_1x_0^2 - x_0x_2x_0x_1x_0^2) \qquad (18)\\
&\quad \cdot (1 - x_2x_0x_1x_0^2 + x_0x_2x_0x_1x_0^2\\
&\quad + x_0 - x_0 + x_0x_2x_0x_1x_0^2 - x_0^2 + x_0^2)\\
&= (1 + x_2x_0x_1x_0^2 - x_0x_2x_0x_1x_0^2)\\
&\quad \cdot (1 - x_2x_0x_1x_0^2 + 2x_0x_2x_0x_1x_0^2)\\
&= (1 + x_0x_2x_0x_1x_0^2).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
[z, 1 + x_0, 1 + x_0, 1 + x_0] &= (1 + x_0x_2x_0x_1x_0^2)^{-1}(1 + x_0)^{-1}\\
&\quad (1 + x_0x_2x_0x_1x_0^2)(1 + x_0)\\
&= (1 - x_0x_2x_0x_1x_0^2)(1 - x_0 + x_0^2)\\
&\quad (1 + x_0x_2x_0x_1x_0^2 + x_0)\\
&= (1 - x_0x_2x_0x_1x_0^2)\\
&\quad (1 + x_0x_2x_0x_1x_0^2 + x_0 - x_0 - x_0^2 + x_0^2)\\
&= (1 - x_0x_2x_0x_1x_0^2)\\
&\quad (1 + x_0x_2x_0x_1x_0^2)\\
&= 1.
\end{aligned}
$$

This completes the proof. □

1.3. Computer Work. This subsection provides a presentation for the Gupta and Levin example that is in a convenient form for computer calculations. We will call the group of the Gupta and Levin example H, while the group we construct on the computer we will call G. Consider the group G with the presentation $\langle x, y, z \mid [y, z] = [x, y, y] = [x, z, z] = [x, y, z, x, x, x] = [x, y, z][x, z, y] = w^5 = 1, \text{for all words } w \text{ in } x, y, z\rangle$. The relators in G are relators that hold in H by Lemma 3.7, so H is a quotient of G. The group G can be entered into GAP with the code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=["[f2,f3]",
```

```
"[f1,f2,f2]",
"[f1,f3,f3]",
"[f1,f2,f3,f1,f1,f1]",
"[f1,f2,f3]*[f1,f3,f2]"];
g:=Pq(f: Prime:=5, Exponent:=5, Relators:=rels);
```

When we create G in GAP, it gives the output

```
#I  Class 1 with 3 generators.
#I  Class 2 with 5 generators.
#I  Class 3 with 7 generators.
#I  Class 4 with 10 generators.
#I  Class 5 with 11 generators.
#I  Class 6 with 12 generators.
```

From this output, we read that G has pc-structure $\{3, 5, 7, 10, 11, 12\}$. In addition, GAP says:

```
gap> g;
<pc group of size 244140625 with 12 generators>
gap> Factors(Order(g));
[ 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5 ]
```

This output tells us that G is of order 5^{12}.

Because all of the relators and laws we impose in G are known to hold in H, we know that H is a quotient of G. We have determined using GAP that G is of order 5^{12}, so H has order less than or equal to 5^{12}, not 5^{20} as shown earlier. Table 2 gives a (not necessarily unique) listing of pc generators for G, using simple commutators for consistency with later work. The weight 1 pc-generators are the same as the group generators. For higher weights, GAP was used to find a simple commutator that GAP reported was a pc-generator. For instance, when the commutator $[x, y]$ was entered into GAP, it returned $g.4$, so we know that $[x, y]$ can be considered the fourth pc-generator of G.

2. Restrictions on Groups with Condition $\mathfrak{E}_4$ but not Condition $\mathfrak{L}(\mathfrak{N}_3)$

We know that the Gupta and Levin example H has order at most 5^{12}. We also know that H is a 4-Engel group and that it has an element x with a normal closure

Table 2: PC-generators for G

x	y	z
$[x, y]$	$[x, z]$	
$[x, y, x]$	$[x, z, x]$	
$[x, y, x, x]$	$[x, z, x, x]$	$[x, y, x, z]$
$[x, y, x, x, z]$		
$[x, y, x, x, z, x]$		

x^H that is not nilpotent of class 3. We say that a group K has property $\mathfrak{T}$ if K is 4-Engel and there is some element x of K such that the normal closure x^K is not nilpotent of class 3. Traustason has shown that a group with property $\mathfrak{T}$ must have at least one element of order 2 or of order 5 [**47**]. In this section, we find some limitations on groups that have property $\mathfrak{T}$. The goal of this section is to prove the following theorems:

THEOREM 3.8. *If G is a 4-Engel group and is nilpotent of class at most 5, then the normal closure in G of every element of G is nilpotent of class at most 3.*

THEOREM 3.9. *If G is a 4-Engel group on two generators with exponent 5, then for every element a in G, the normal closure a^G is nilpotent of class at most 3.*

2.1. Nilpotence Class Restrictions. We begin by showing that a group with property $\mathfrak{T}$ must not be nilpotent of class 5 or smaller.

PROOF OF THEOREM 3.8. Let x, a, b, c, d be elements of G. We want to show that x^G is nilpotent of class at most 3. It suffices to show that a commutator of weight 4 in the generators of x^G, in this case $[x^a, x^b, x^c, x^d]$, is trivial. First we calculate

$$\begin{aligned}
[x^a, x^b]\gamma_4(G) &= [(x^b)^{(b^{-1}a)}, x^b]\gamma_4(G) \\
&= [(b^{-1}a)^{-1}x^b(b^{-1}a), x^b]\gamma_4(G) \\
&= [(b^{-1}a)^{-1}x^b, x^b]^{(b^{-1}a)}[b^{-1}a, x^b]\gamma_4(G) \\
&= ([(b^{-1}a)^{-1}, x^b]^{x^b}[x^b, x^b])^{b^{-1}a}[b^{-1}a, x^b]\gamma_4(G)
\end{aligned}$$

$$
\begin{aligned}
&= [(b^{-1}a)^{-1}, x^b]^{(x^b b^{-1}a)}[b^{-1}a, x^b]\gamma_4(G) \qquad (19)\\
&= ([b^{-1}a, x^b]^{-1})^{(b^{-1}a)^{-1}x^b b^{-1}a}[b^{-1}a, x^b]\gamma_4(G)\\
&= [b^{-1}a, x^b, (b^{-1}a)^{-1}x^b(b^{-1}a)]^{-1}\gamma_4(G)\\
&= [b^{-1}a, x^b, [b^{-1}a, x^{-b}]x^b)]^{-1}\gamma_4(G)\\
&= ([b^{-1}a, x^b, x^b][b^{-1}a, x^b, [b^{-1}a, x^{-b}]]^{x^b})^{-1}\gamma_4(G)\\
&= [b^{-1}a, x^b, x^b]^{-1}\gamma_4(G).
\end{aligned}
$$

We have just shown that $[x^a, x^b] = [b^{-1}a, x^b, x^b]^{-1}y$, where y is in $\gamma_4(G)$, so

$$
\begin{aligned}
[x^a, x^b, x^c] &= [[b^{-1}a, x^b, x^b]^{-1}y, x^c] \qquad (20)\\
&= [b^{-1}a, x^b, x^b]^{-1}, x^c][b^{-1}a, x^b, x^b]^{-1}, x^c, y][y, x^c]. \qquad (21)
\end{aligned}
$$

Because $[[b^{-1}a, x^b, x^b]^{-1}, x^c, y]$ and $[y, x^c]$ are both in $\gamma_5(G)$,

$$
\begin{aligned}
[x^a, x^b, x^c]\gamma_5(G) &= [[b^{-1}a, x^b, x^b]^{-1}, x^c]\gamma_5(G)\\
&= ([b^{-1}a, x^b, x^b, x^c]^{-1})^{[b^{-1}a, x^b, x^b]^{-1}}\gamma_5(G)\\
&= ([b^{-1}a, x^b, x^b, x^c]^{-1})\gamma_5(G)\\
&= [b^{-1}a, x^b, x^b, (x^b)^{b^{-1}c}]^{-1}\gamma_5(G) \qquad (22)\\
&= [b^{-1}a, x^b, x^b, [b^{-1}c, (x^b)^{-1}]x^b]^{-1}\gamma_5(G)\\
&= ([b^{-1}a, x^b, x^b, x^b][b^{-1}a, x^b, x^b, [b^{-1}c, (x^b)^{-1}]]^{x^b})^{-1}\gamma_5(G)\\
&= [b^{-1}a, x^b, x^b, x^b]^{-1}\gamma_5(G).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
[x^a, x^b, x^c, x^d]\gamma_6(G) &= [[b^{-1}a, x^b, x^b, x^b]^{-1}, x^d]\gamma_6(G)\\
&= ([b^{-1}a, x^b, x^b, x^b, x^d]^{-1})^{[b^{-1}a, x^b, x^b, x^b]^{-1}}\gamma_6(G)\\
&= [b^{-1}a, x^b, x^b, x^b, x^d]^{-1}\gamma_6(G)\\
&= [b^{-1}a, x^b, x^b, x^b, (x^b)^{b^{-1}d}]^{-1}\gamma_6(G) \qquad (23)\\
&= [b^{-1}a, x^b, x^b, x^b, [b^{-1}d, (x^b)^{-1}]x^b]^{-1}\gamma_6(G)
\end{aligned}
$$

$$
\begin{aligned}
&= ([b^{-1}a, x^b, x^b, x^b, x^b][b^{-1}a, x^b, x^b, x^b, [b^{-1}d, (x^b)^{-1}]]^{x^b})^{-1}\gamma_6(G) \\
&= [b^{-1}a, x^b, x^b, x^b, x^b]^{-1}\gamma_6(G).
\end{aligned}
$$

Since G is 4-Engel, we conclude that $[x^a, x^b, x^c, x^d]$ is in $\gamma_6(G)$. Since G is nilpotent of class at most 5, we conclude that x^G is nilpotent of class at most 3. $\square$

Hence, an $\mathfrak{T}$ group, if nilpotent, must be nilpotent of class at least 6.

2.2. Two Generators. In this subsection we prove Theorem 3.9.

Proof of Theorem 3.9. Let G be a 4-Engel group on generators a and b with exponent 5. Let $\widehat{G}$ be the group given by the presentation $\langle a, b \mid w^5 = [w, u, u, u, u] = 1$ for all w and u words in a and $b\rangle$. Note that G is a quotient of $\widehat{G}$. We will show that $\widehat{G}$ has nilpotence class 6.

If we give GAP the code:

```
f:=FreeGroup(2);
g:=Pq(f: Prime:=5, Exponent:=5, Identities:=[function(a,b)
return LeftNormedComm([a,b,b,b,b]);end]);
```

then GAP returns the answer:

```
#I  Class 1 with 2 generators.
#I  Class 2 with 3 generators.
#I  Class 3 with 5 generators.
#I  Class 4 with 8 generators.
#I  Class 5 with 10 generators.
#I  Class 6 with 11 generators.
#I  Class 6 with 11 generators.
```

This output shows that $\widehat{G}$ is nilpotent of class 6, so G is also nilpotent of class 6.

Now we consider the commutator $[x, x^c, x^d, x^e]$, where x, c, d, and e are in G. Then

$$
\begin{aligned}
[x, x^c, x^d, x^e] &= [x, x[x, c], x^d, x^e] \\
&= [[x, [x, c]][x, x]^{[x,c]}, x^d, x^e] \\
&= [[x, [x, c]], x^d, x^e] \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
&= [[x,[c,x]^{-1}],x^d,x^e] \\
&= [[c,x,x]^{[c,x]^{-1}},x^d,x^e] \\
&= [[c,x,x][[c,x,x],[c,x]^{-1}],x^d,x^e].
\end{aligned}
$$

Because G is nilpotent of class at most 6 and $[[c,x,x],[c,x]^{-1}]$ is in $\gamma_5(G)$,

$$
\begin{aligned}
[x,x^c,x^d,x^e] &= [c,x,x,x^d,x^e] \\
&= [c,x,x,d^{-1}(xd),x^e] \\
&= [[c,x,x,xd][c,x,x,d^{-1}][c,x,x,d^{-1},xd],x^e] \\
&= [[c,x,x,d][c,x,x,x][c,x,x,x,d][c,x,x,d^{-1}][c,x,x,d^{-1},d] \\
&\qquad [c,x,x,d^{-1},x][c,x,x,d^{-1},x,d],x^e].
\end{aligned}
\tag{25}
$$

Weight 4 and larger commutators in G commute with each other because G is nilpotent of class at most 6. Thus,

$$
\begin{aligned}
[x,x^c,x^d,x^e] = [[&c,x,x,d][c,x,x,d^{-1}][c,x,x,d^{-1},d][c,x,x,x][c,x,x,x,d] \\
&[c,x,x,d^{-1},x][c,x,x,d^{-1},x,d],x^e].
\end{aligned}
$$

Now we note that $[c,x,x,d][c,x,x,d^{-1}][c,x,x,d^{-1},d] = [c,x,x,x,d^{-1}d] = 1$, so

$$
[x,x^c,x^d,x^e] = [[c,x,x,x][c,x,x,x,d][c,x,x,d^{-1},x][c,x,x,d^{-1},x,d],x^e]. \tag{26}
$$

Because G is nilpotent of class 6 and $[c,x,x,d^{-1},x,d]$ is in $\gamma_6(G)$, we see that

$$
\begin{aligned}
[x,x^c,x^d,x^e] &= [[c,x,x,x][c,x,x,x,d][c,x,x,d^{-1},x],x^e] \\
&= [c,x,x,x,x^e]^{[c,x,x,x,d][c,x,x,d^{-1},x]} \\
&\qquad [[c,x,x,x,d][c,x,x,d^{-1},x],x^e] \\
&= [c,x,x,x,x^e]^{[c,x,x,x,d][c,x,x,d^{-1},x]} \\
&\qquad [c,x,x,x,d,x^e]^{[c,x,x,d^{-1},x]}[c,x,x,d^{-1},x,x^e].
\end{aligned}
\tag{27}
$$

Since G is nilpotent of class at most 6,

$$[x, x^c, x^d, x^e] = [c, x, x, x, x^e][c, x, x, x, d, x^e][c, x, x, d^{-1}, x, x^e]. \tag{28}$$

Consider the first factor:

$$\begin{aligned}[c, x, x, x, x^e] &= [c, x, x, x, xe][c, x, x, x, e^{-1}][c, x, x, x, e^{-1}, xe] \\ &= [c, x, x, x, e][c, x, x, x, x][c, x, x, x, x, e][c, x, x, x, e^{-1}] \\ &\qquad [c, x, x, x, e^{-1}, e][c, x, x, x, e^{-1}, x][c, x, x, x, e^{-1}, x, e].\end{aligned} \tag{29}$$

Because G is 4-Engel, the second and third factors of (29) are trivial, and the seventh factor is trivial because G is nilpotent of class at most 6. Thus,

$$\begin{aligned}[c, x, x, x, x^e] &= [c, x, x, x, e][c, x, x, x, e^{-1}][c, x, x, x, e^{-1}, e] \\ &\qquad [c, x, x, x, e^{-1}, x] \\ &= [c, x, x, x, ee^{-1}][c, x, x, x, e^{-1}, x] \\ &= [c, x, x, x, e^{-1}, x].\end{aligned} \tag{30}$$

Combining (30) with (28),

$$[x, x^c, x^d, x^e] = [c, x, x, x, e^{-1}, x][c, x, x, x, d, x^e][c, x, x, d^{-1}, x, x^e]. \tag{31}$$

Now, because G is nilpotent of class 6, we use Lemma 1.2 to see that

$$[x, x^c, x^d, x^e] = [c, x, x, x, e^{-1}, x][c, x, x, x, d, x][c, x, x, d^{-1}, x, x]. \tag{32}$$

The element x can be written as $a^i b^j z$, where z is in $\gamma_2(G)$ and $0 \leq i, j \leq 4$. Then

$$\begin{aligned}[x, x^c, x^d, x^e] &= [c, x, x, x, e^{-1}, x][c, x, x, x, d, x][c, x, x, d^{-1}, x, x] \\ &= [c, a^i b^j z, a^i b^j z, a^i b^j z, e^{-1}, a^i b^j z][c, a^i b^j z, a^i b^j z, a^i b^j z, d, a^i b^j z] \\ &\qquad \cdot [c, a^i b^j z, a^i b^j z, d^{-1}, a^i b^j z, a^i b^j z].\end{aligned} \tag{33}$$

Now,

$$[c,x]\gamma_3(G) = [c,a^ib^jz]\gamma_3(G)$$
$$= [c,z][c,a^ib^j][c,a^ib^j,z]\gamma_3(G) \tag{34}$$
$$= [c,z][c,a^ib^j]\gamma_3(G).$$

Because z is in $\gamma_2(G)$, the commutator $[c,z]$ is in $\gamma_3(G)$, so

$$[c,x]\gamma_3(G) = [c,a^ib^j]\gamma_3(G). \tag{35}$$

Similarly,

$$[c,x,x]\gamma_4(G) = [c,a^ib^j,a^ib^j]\gamma_4(G). \tag{36}$$

Moreover,

$$[c,x,x,x]\gamma_5(G) = [c,a^ib^j,a^ib^j,a^ib^j]\gamma_5(G) \tag{37}$$

and

$$[c,x,x,d^{-1}]\gamma_5(G) = [c,a^ib^j,a^ib^j,d^{-1}]\gamma_5(G). \tag{38}$$

Further,

$$[c,x,x,x,e^{-1}]\gamma_6(G) = [c,a^ib^j,a^ib^j,a^ib^j,e^{-1}]\gamma_6(G), \tag{39}$$

$$[c,x,x,x,d]\gamma_6(G) = [c,a^ib^j,a^ib^j,a^ib^j,d]\gamma_6(G), \tag{40}$$

and

$$[c,x,x,d^{-1},x]\gamma_6(G) = [c,a^ib^j,a^ib^j,d^{-1},a^ib^j]\gamma_6(G). \tag{41}$$

Finally,

$$[c,x,x,x,e^{-1},x]\gamma_7(G) = [c,a^ib^j,a^ib^j,a^ib^j,e^{-1},a^ib^j]\gamma_7(G), \tag{42}$$

$$[c,x,x,x,d,x]\gamma_7(G) = [c,a^ib^j,a^ib^j,a^ib^j,d,a^ib^j]\gamma_7(G), \tag{43}$$

and

$$[c,x,x,d^{-1},x,x]\gamma_7(G) = [c,a^ib^j,a^ib^j,d^{-1},a^ib^j,a^ib^j]\gamma_7(G). \tag{44}$$

Because G is nilpotent of class at most 6, we see $\gamma_7(G)$ is trivial, so

$$[x, x^c, x^d, x^e] = [c, a^ib^j, a^ib^j, a^ib^j, e^{-1}, a^ib^j][c, a^ib^j, a^ib^j, a^ib^j, d, a^ib^j] \cdot [c, a^ib^j, a^ib^j, d^{-1}, a^ib^j, a^ib^j]. \tag{45}$$

Now (32) holds for all x in G, including a^ib^j, so

$$[x, x^c, x^d, x^e] = [a^ib^j, (a^ib^j)^c, (a^ib^j)^d, (a^ib^j)^e]. \tag{46}$$

We conclude that we may assume that z is trivial.

Now we can use GAP to show that the normal closure of every element in $\widehat{G}$ of the form a^ib^j, where $0 \leq i, j \leq 4$, is nilpotent of class at most 3. This result is seen by the following output from GAP:

```
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^3])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^4])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1*g.2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^2*g.2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^3*g.2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1^4*g.2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.2^2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1*g.2^2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^2*g.2^2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^3*g.2^2])));
```

```
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^4*g.2^2])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.2^3])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1*g.2^3])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^2*g.2^3])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^3*g.2^3])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^4*g.2^3])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.2^4])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1*g.2^4])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^2*g.2^4])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^3*g.2^4])));
3
gap> NilpotencyClassOfGroup(NormalClosure(g,Subgroup
(g,[g.1^4*g.2^4])));
3
```

Thus, the normal closure of every element of $\widehat{G}$ is nilpotent of class at most 3. Because G is a quotient of $\widehat{G}$, the normal closure of every element of G is also nilpotent of class at most 3. This completes the proof of Theorem 3.9. $\square$

3. Nickel Counterexample

The example by Gupta and Levin suffices to show that condition $\mathfrak{E}_4$ is not equivalent to condition $\mathfrak{L}(\mathfrak{N}_3)$, but it only provides a counterexample when an element of order 5 is present. In [**5**], Nickel demonstrates how to input in a computer the largest nilpotent quotient of $E(i,n)$, the largest n-Engel group on i generators. One of the groups he uses as an example is $E(3,4)$, for which he gives the nilpotence class, Hirsch length, and structure of the lower central series. In a comment in [**62**], Traustason says that Nickel's paper provides an example of a 4-Engel 2-group that contains an element whose normal closure in the group is not nilpotent of class 3. Using GAP and ANUPQ, we compute the largest quotient of $E(3,4)$ that is nilpotent of class 6 and is a 2-group. The nilpotence class bound is set to 6 because a 4-Engel group that is nilpotent of class 5 will satisfy condition $\mathfrak{L}(\mathfrak{N}_3)$, so we need a class bound of at least 6, and we set it as small as possible to save computer time and produce a smaller group. The following code generates the example that Traustason refers to and computes the nilpotence class of the normal closure of the first generator of the resulting group:

```
RequirePackage("anupq");
f:=FreeGroup(3);
g:=Pq(f: Prime:=2, ClassBound:=6,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

The output from generating this group g in GAP and then asking GAP for more information is the following:

```
#I  Class 1 with 3 generators.
#I  Class 2 with 9 generators.
#I  Class 3 with 23 generators.
#I  Class 4 with 55 generators.
#I  Class 5 with 121 generators.
#I  Class 6 with 212 generators.
gap> c;
4
gap> Order(g);
6582018229284824168619876730229402019930943462534319453394436096
```

As the computer output shows, the normal closure of the first generator of group g has nilpotence class 4 rather than 3, so this group g is in $\mathfrak{E}_4$ but not in $\mathfrak{L}(\mathfrak{N}_3)$. Note that g has order 2^{212}.

4. Smaller 2-group

In this section we construct a much smaller 4-Engel 2-group that does not satisfy Condition $\mathfrak{L}(\mathfrak{N}_3)$ than the example mentioned by Traustason.

THEOREM 3.10. *There is a 4-Engel 2-group of order* 2^{26} *that does not satisfy condition* $\mathfrak{L}(\mathfrak{N}_3)$.

This smaller 2-group example is the group with the presentation $G = \langle x, y, z \mid x^4 = y^2 = z^2 = [y, z, y] = [x, y, z, z] = [y, z, x, x, x] = [x, y, z, x, y] = \gamma_7(\langle x, y, z\rangle) = w^8 = [u, {}_4w] = 1$, for all words u and w in $x, y, z\rangle$. It is created in GAP by the computer code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f1^4",
"f2^2",
"f3^2",
"[f2,f3,f2]",
"[f1,f2,f3,f3]",
"[f2,f3,f1,f1,f1]",
"[f1,f2,f3,f1,f2]",];

G:=Pq(f: Prime:=2, Exponent:=8, ClassBound:=6, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(G,Subgroup(G,[G.1])));
```

The p-class bound is set to 6 because a larger bound would result in a larger group, while a smaller p-class bound would mean the example would satisfy condition $\mathfrak{L}(\mathfrak{N}_3)$. GAP's output from this code is the following:

```
#I  Class 1 with 3 generators.
#I  Class 2 with 7 generators.
#I  Class 3 with 13 generators.
#I  Class 4 with 20 generators.
#I  Class 5 with 25 generators.
#I  Class 6 with 26 generators.
gap> G;
<pc group of size 67108864 with 26 generators>
gap> c;
4
```

This group G has the pc structure $\{3, 7, 13, 20, 25, 26\}$ and order 2^{26}, so it is a 2-group. G is also nilpotent of class 6. The normal closure of the element $G.1$, which is the first generator of G, is nilpotent of class 4, not class 3. These observations complete the proof of Theorem 3.10.

It should be noted that 2^{26} is smaller than 5^{12}, so the smallest known 4-Engel group that does not satisfy condition $\mathfrak{L}(\mathfrak{N}_3)$ is this 2-group G. This example was found by trial and error, as follows. We start with the free group f on three generators. Choosing the exponent to be 2 or 4 gives a group in which the normal closure of $f.1$ is nilpotent of class 3, so the exponent is fixed at 8. We try introducing the possible relator $G.1^2$, and find that this relator makes the nilpotence class of the normal closure of $G.1$ be 3. The relator $G.1^4$ does not give nilpotence class 3 for the normal closure of $G.1$, so we leave it among the relators. Then we try $G.2^2$, and again the nilpotence class of the normal closure of $G.1$ is larger than 3. The possible relator $G.3^2$ gives the same result. Next a possible weight two commutator is tested. The possible weight two commutators are $[f.1, f.2]$, $[f.1, f.3]$, and $[f.2, f.3]$. Introducing any of these weight two commutators again gives a nilpotence class of 3 for the normal closure of $G.1$, so we do not include any weight two relators in the group G. Then we consider all weight three commutators in the generators of G. Again, we try the possible relators of weight three one at a time, and keep those that do not shrink the nilpotence class of the normal closure of $G.1$ but do make the pc-structure smaller. We repeat this process for possible relators of weight 4, 5, and 6. While we have not proved that the group G so constructed is the smallest example of a 4-Engel group that has an element whose normal closure is nilpotent of class exactly 4, we believe that it is close to being the smallest such group.

5. Larger n and Other Primes

Similar experimentation produces other examples of n-Engel groups that contain an element whose normal closure is not nilpotent of class $n-1$, that is, do not satisfy Condition $\mathfrak{L}(\mathfrak{N}_{n-1})$, for larger values of n. In each of the groups discussed in this section, the normal closure of first generator is not nilpotent of class $n-1$.

The general method for constructing the examples in this section is to start with a group with the lowest possible nilpotence class c and least exponent that is n-Engel, and check if the normal closure of the first generator is nilpotent of class $c-1$. If so, first weight two commutators are added as relators to make the second part of the pc structure as small as possible. Then weight three commutators are considered, and the process is repeated until the weight reaches the p-class. This procedure might not produce for each p the smallest p-group that has condition $\mathfrak{E}_n$ but not condition $\mathfrak{L}(\mathfrak{N}_{n-1})$ for the desired integer n, but it does produce reasonably small groups. The examples in this section prove the following two theorems.

THEOREM 3.11. *There exist 5-Engel and 6-Engel 5-groups that do not satisfy conditions* $\mathfrak{L}(\mathfrak{N}_4)$ *and* $\mathfrak{L}(\mathfrak{N}_5)$ *respectively. There also exist 5-Engel and 6-Engel 2-groups that do not satisfy conditions* $\mathfrak{L}(\mathfrak{N}_4)$ *and* $\mathfrak{L}(\mathfrak{N}_5)$ *respectively.*

Hence condition $\mathfrak{E}_5$ does not imply condition $\mathfrak{L}(\mathfrak{N}_4)$ and condition $\mathfrak{E}_6$ does not imply condition $\mathfrak{L}(\mathfrak{N}_5)$.

Theorem 3.11 is new, but Vaughan-Lee [**4**] exhibits a 5-group that is 5-Engel but that has a normal closure that is not 4-Engel, so that normal closure cannot be nilpotent of class 4. Vaughan-Lee's example is given in detail in Section 1 of Chapter 5.

THEOREM 3.12. *There exist 5-Engel 3-groups and 7-groups that do not satisfy condition* $\mathfrak{L}(\mathfrak{N}_4)$.

Theorem 3.12 shows that for 3-groups and 7-groups there exists an m, namely $m=5$, such that condition $\mathfrak{E}_m$ does not imply condition $\mathfrak{L}(\mathfrak{N}_{m-1})$. We show how to construct examples that justify Theorems 3.11 and 3.12 in the next two subsections.

5.1. 5-Engel groups. A reasonably small $\mathfrak{E}_5$ 5-group G_5 without $\mathfrak{L}(\mathfrak{N}_4)$ is given by the presentation $G_5 = \langle x, y, z \mid [y,z] = [x,y,y] = [x,y,z] = [x,z,z] =$

$\gamma_8(\langle x, y, z\rangle) = w^5 = 1$ for all words w in $x, y, z\rangle$. This group is nilpotent of class 7, has the pc structure $\{3, 5, 7, 10, 11, 12, 13\}$, and has order 5^{13}. The GAP code for this group is

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"[f2,f3]",
"[f1,f2,f2]",
"[f1,f2,f3]",
"[f1,f3,f3]"];
g:=Pq(f: Prime:=5, Exponent:=5, ClassBound:=7, Relators:=rels);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

GAP's output for G_5 is the following:

```
gap> g;
<pc group of size 1220703125 with 13 generators>
gap> c;
5
```

G_5 is 5-Engel because the quotient of G_5 obtained by adding the 5-Engel law has the same order as G_5.

A reasonably small $\mathfrak{E}_5$ 2-group G_2 without $\mathfrak{L}(\mathfrak{N}_4)$ has the presentation $G_2 = \langle x, y, z \mid x^4 = y^2 = z^2 = [y, z, z] = [x, y, y] = [x, z, x]^2 = [y, z, x]^2 = [x, y, x, x] = [x, y, z, y] = [x, y, z, z] = [x, z, y, y] = [x, z, z, x] = [y, z, x, y] = [x, y, x, z, z] = [x, y, z, x, y] = [x, z, y, x, z] = [x, z, z, y, z] = [y, z, x, x, x] = [x, y, x, z, x, z] = [x, y, z, x, x, z] = \gamma_8(\langle x, y, z\rangle) = [u, {}_5w] = w^8 = 1$, for all u and w words in $x, y, z\rangle$. The group G_2 is input in GAP with the code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f1^4",
"f2^2",
"f3^2",
"[f2,f3,f3]",
"[f1,f2,f2]",
"[f1,f3,f1]^2",
"[f2,f3,f1]^2",
"[f1,f2,f1,f1]",
```

```
"[f1,f2,f3,f2]",
"[f1,f2,f3,f3]",
"[f1,f3,f2,f2]",
"[f1,f3,f3,f1]",
"[f2,f3,f1,f2]",
"[f1,f2,f1,f3,f3]",
"[f1,f2,f3,f1,f2]",
"[f1,f3,f2,f1,f3]",
"[f1,f3,f3,f2,f3]",
"[f2,f3,f1,f1,f1]",
"[f1,f2,f1,f3,f1,f3]",
"[f1,f2,f3,f1,f1,f3]"];
g:=Pq(f: Prime:=2, Exponent:=8, ClassBound:=7, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

Here the p-class bound is set to 7 because in a group that is nilpotent of class 6 the normal closure of every element in the group is nilpotent of class 5, and the nilpotence and p-class bounds usually match up rather well in the groups we examine. The exponent is set to 8 because a smaller exponent does not give the desired nilpotence class for a normal closure. GAP gives the following output for this code:

```
#I  Class 1 with 3 generators.
#I  Class 2 with 7 generators.
#I  Class 3 with 12 generators.
#I  Class 4 with 17 generators.
#I  Class 5 with 21 generators.
#I  Class 6 with 23 generators.
#I  Class 7 with 24 generators.
gap> g;
<pc group of size 16777216 with 24 generators>
gap> Order(g);
16777216
gap> NilpotencyClassOfGroup(g);
7
gap> c;
5
```

Note that G_2 has order 2^{24}.

5.2. 6-Engel Groups. The group given to GAP by the code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f1^5",
"f2^5",
"f3^5",
"[f2,f3]",
"[f1,f2,f2]",
"[f1,f2,f3]",
"[f1,f3,f3]",
"[f1,f2,f1,f2]",
"[f1,f3,f1,f3]",
"[f1,f3,f2,f2]",
"[f1,f2,f1,f1,f1]",
"[f1,f2,f1,f3,f3]",
"[f1,f3,f1,f1,f1]"];
g:=Pq(f: Prime:=5, Exponent:=25, ClassBound:=8, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

is the 6-Engel group H_5 given by

$\langle x, y, z \mid x^5 = y^5 = z^5 = [y, z] = [x, y, y] = [x, y, z] = [x, z, z] = [x, y, x, y] = [x, z, x, z] = [x, z, y, y] = [x, y, x, x, x] = [x, y, x, z, z] = [x, z, x, x, x] = \gamma_8(\langle x, y, z\rangle) = [w, {}_6u] = w^{25} = 1$ for all words w, u in $x, y, z\rangle$. Then H_5 is an $\mathfrak{E}_6$ 5-group of order 5^{15}, but not a $\mathfrak{L}(\mathfrak{N}_5)$ group, as is seen by the output:

```
#I  Class 1 with 3 generators.
#I  Class 2 with 5 generators.
#I  Class 3 with 7 generators.
#I  Class 4 with 10 generators.
#I  Class 5 with 12 generators.
#I  Class 6 with 13 generators.
#I  Class 7 with 14 generators.
#I  Class 8 with 15 generators.
gap> g;
<pc group of size 30517578125 with 15 generators>
gap> c;
6
```

This group H_5 is not of exponent 5, because if we have GAP compute the exponent 5 quotient of H_5, the resulting group has smaller order.

Similarly, the 6-Engel 2-group H_2 given by the GAP code

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f1^4",
"f2^2",
"f3^2",
"[f2,f3]",
"[f1,f2,f3]",
"[f1,f2,f1,f2]",
"[f1,f2,f2,f2]",
"[f1,f3,f1,f3]",
"[f1,f3,f3,f2]",
"[f1,f3,f3,f3]",
"[f1,f2,f2,f1,f3]",
"[f1,f3,f2,f1,f1,f1]",
];
g:=Pq(f: Prime:=2, Exponent:=8, ClassBound:=8, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

has presentation $\langle x, y, z \mid x^4 = y^2 = z^2 = [y, z] = [x, y, z] = [x, y, x, y] = [x, y, y, y] = [x, z, x, z] = [x, z, z, y] = [x, z, z, z] = [x, y, y, x, z] = [x, z, y, x, x, x] = \gamma_8(\langle x, y, z\rangle) = w^8 = [u, {}_6 w] = 1$ for all words u and w in $x, y, z\rangle$.

GAP gives the output:

```
#I  Class 1 with 3 generators.
#I  Class 2 with 6 generators.
#I  Class 3 with 10 generators.
#I  Class 4 with 13 generators.
#I  Class 5 with 17 generators.
#I  Class 6 with 21 generators.
#I  Class 7 with 22 generators.
#I  Class 8 with 23 generators.
gap> g;
<pc group of size 8388608 with 23 generators>
gap> Order(g);
8388608
gap> NilpotencyClassOfGroup(g);
8
gap> c;
6
```

This group H_2 has order 2^{23}, and does not satisfy $\mathfrak{L}(\mathfrak{N}_5)$. The examples just discussed in subsections 5.1 and 5.2 justify Theorem 3.11.

5.3. Other Primes. As noted earlier, it is known that a group must have an element of order 2 or 5 to be 4-Engel but contain an element whose normal closure is not nilpotent of class 3, but it is shown here that the same restriction does not hold for 5-Engel groups that contain an element whose normal closure is not nilpotent of class 4. A reasonably small 5-Engel 3-group G_3 of order 3^{12} that contains an element whose normal closure in not nilpotent of class 4 is given by GAP code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"f2^3",
"f3^3",
"[f2,f3]",
"[f1,f2,f1]",
"[f1,f2,f2]",
"[f1,f2,f3]",
"[f1,f3,f3]"];
g:=Pq(f: Prime:=3, Exponent:=9, ClassBound:=7, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

and the presentation $\langle x, y, z \mid y^3 = z^3 = [y, z] = [x, y, x] = [x, y, y] = [x, y, z] = [x, z, z] = \gamma_8(\langle x, y, z\rangle) = w^9 = [u, {}_5w] = 1$ for all words u and w in $x, y, z\rangle$.

GAP gives as output:

```
#I  Class 1 with 3 generators.
#I  Class 2 with 6 generators.
#I  Class 3 with 7 generators.
#I  Class 4 with 8 generators.
#I  Class 5 with 10 generators.
#I  Class 6 with 11 generators.
#I  Class 7 with 12 generators.
gap> g;
<pc group of size 531441 with 12 generators>
gap> Order(g);
531441
gap> NilpotencyClassOfGroup(g);
```

```
7
gap> c;
5
```

This 5-Engel group G_3 of order 3^{12} does not satisfy $\mathfrak{L}(\mathfrak{E}_4)$. A $\mathfrak{E}_5$ 7-group G_7 of order 7^{16} can be obtained by the GAP code:

```
RequirePackage("anupq");
f:=FreeGroup(3);
rels:=[
"[f2,f3]",
"[f1,f2,f2]",
"[f1,f2,f3]",
"[f1,f3,f3]",
"[f1,f2,f1,f2]",
"[f1,f3,f1,f3]"];
g:=Pq(f: Prime:=7, Exponent:=7, ClassBound:=7, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
end]);
c:=NilpotencyClassOfGroup(NormalClosure(g,Subgroup(g,[g.1])));
```

and is given by the presentation

$\langle x, y, z, \mid [y, z] = [x, y, y] = [x, y, z] = [x, z, z] = [x, y, x, y] = [x, z, x, z] =$

$\gamma_8(\langle x, y, z\rangle) = w^7 = [u, {}_5w] = 1$ for all words u and w in $x, y, z\rangle$. When this code is input into GAP, the results are:

```
#I  Class 1 with 3 generators.
#I  Class 2 with 5 generators.
#I  Class 3 with 7 generators.
#I  Class 4 with 10 generators.
#I  Class 5 with 14 generators.
#I  Class 6 with 15 generators.
#I  Class 7 with 16 generators.
gap> g;
<pc group of size 33232930569601 with 16 generators>
gap> Order(g);
33232930569601
gap> NilpotencyClassOfGroup(g);
7
gap> c;
5
```

This 5-Engel group G_7 of order 7^{16} does not satisfy $\mathfrak{L}(\mathfrak{E}_4)$.

The examples in subsection 5.3 justify Theorem 3.12.

Chapter 4

Examples Where $\mathfrak{E}_n$ Does Not Imply $\mathfrak{L}(\mathfrak{N})$

The examples in Chapter 3 show that a group can satisfy condition $\mathfrak{E}_n$, the condition of being an n-Engel group, but not satisfy condition $\mathfrak{L}(\mathfrak{N}_{n-1})$, meaning they contain an element whose normal closure is not nilpotent of class 3. However, the examples in Chapter 3 all satisfy $\mathfrak{L}(\mathfrak{N})$, the condition of every normal closure being nilpotent, because the groups are nilpotent. Gupta and Levin in [**3**] give a family of examples that show that there exist groups that are n-Engel but in which there is an element with a non-nilpotent normal closure in the group. We follow their construction.

Construction 1 (Construction of M_p). *Let p be an odd prime. Let G_p be the relatively free group on countably many generators $\{x_0, x_1, \dots\}$ with exponent p and of nilpotence class 2. Let M_p be the set of 2×2 matrices over the group ring $\mathbb{Z}_p G_p$ of the form $\left(\begin{smallmatrix} g & 0 \\ r & 1 \end{smallmatrix}\right)$, where g is in G_p and r is in $\mathbb{Z}_p G_p$.*

Lemma 4.1. *(Theorem 1 of Gupta and Levin [**3**]) M_p has the following properties:*

(1) *M_p is a group.*
(2) *M_p has exponent p^2.*
(3) *$\gamma_i(M_p)$ is contained in $\left\{ \left(\begin{smallmatrix} \gamma_i(G_p) & 0 \\ \mathbb{Z}_p G_p & 1 \end{smallmatrix} \right) \right\}$ for each positive integer i.*
(4) *M_p is $(p+2)$-Engel.*
(5) *M_p is not in $\mathfrak{L}(\mathfrak{N})$.*

Proof. (1) Because the set of all invertible 2×2 matrices over $\mathbb{Z}_p G_p$ is a group, we need to show that the set M_p is closed under multiplication and inversion.

To show that M_p is closed under multiplication, let g_1 and g_2 be elements of G and let r_1 and r_2 be elements of $\mathbb{Z}_pG_p$. Then

$$\begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix} = \begin{pmatrix} g_1g_2 & 0 \\ r_1g_2 + r_2 & 1 \end{pmatrix} \in M_p. \tag{47}$$

Also,

$$\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix} \cdot \begin{pmatrix} g^{-1} & 0 \\ -rg^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{48}$$

and

$$\begin{pmatrix} g^{-1} & 0 \\ -rg^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{49}$$

so

$$\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}^{-1} = \begin{pmatrix} g^{-1} & 0 \\ -rg^{-1} & 1 \end{pmatrix} \in M_p. \tag{50}$$

Thus M_p is a group.

(2) To show that the exponent of M_p is p^2, we consider the subset N_p of M_p defined to be the matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right)$, where $r \in \mathbb{Z}_pG_p$. We notice that

$$\begin{pmatrix} 1 & 0 \\ r_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ r_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r_1 + r_2 & 1 \end{pmatrix}, \tag{51}$$

so N_p is closed under multiplication. Also,

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}, \tag{52}$$

so N_p is closed under inversion and thus is a subgroup of M_p. We also observe that N_p is abelian because $r_1 + r_2 = r_2 + r_1$ for all $r_1, r_2 \in \mathbb{Z}_pG_p$. From (47) we see that

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}^{p} = \begin{pmatrix} 1 & 0 \\ pr & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{53}$$

so N_p is of exponent p. Again using (47) we see that

$$\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}^p = \begin{pmatrix} g^p & 0 \\ \hat{r} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \hat{r} & 1 \end{pmatrix}, \tag{54}$$

where $\hat{r} = r \sum_{i=0}^{p-1} g^i$ is in $\mathbb{Z}_p G_p$. Therefore the pth power of any element of M_p is in N_p, and hence M_p is of exponent p^2.

(3) Because the remaining properties are commutator properties, we first calculate a general commutator in M_p. If g_1 and g_2 are elements of G_p and r_1 and r_2 are elements of $\mathbb{Z}_p G_p$, then

$$\begin{aligned}\left[\begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix}, \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix}\right] &= \begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix} \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} g_1^{-1} & 0 \\ -r_1 g_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_2^{-1} & 0 \\ -r_2 g_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ r_1 & 1 \end{pmatrix} \begin{pmatrix} g_2 & 0 \\ r_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} g_1^{-1} g_2^{-1} & 0 \\ -r_1 g_1^{-1} g_2^{-1} - r_2 g_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_1 g_2 & 0 \\ r_1 g_2 + r_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} [g_1, g_2] & 0 \\ -r_1 [g_1, g_2] - r_2 g_1^{g_2} + r_1 g_2 + r_2 & 1 \end{pmatrix}.\end{aligned} \tag{55}$$

By induction, we see that $\gamma_i(M_p)$ is contained in $\left(\begin{smallmatrix} \gamma_i(G_p) & 0 \\ \mathbb{Z}_p G_p & 1 \end{smallmatrix}\right)$.

(4) Let A and B be elements of M_p. The commutator $[A, {}_2B]$ is in $\gamma_3(M_p)$. From (3), it follows that $[A, {}_2B]$ is in $\left(\begin{smallmatrix} \gamma_3(G_p) & 0 \\ \mathbb{Z}_p G_p & 1 \end{smallmatrix}\right)$. However, G_p is nilpotent of class 2, so $[A, {}_2B]$ is in N_p, and hence has the form $\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right)$ for s in $\mathbb{Z}_p G_p$. The matrix B has the form $\left(\begin{smallmatrix} g & 0 \\ r & 1 \end{smallmatrix}\right)$, where g is in G_p and r is in $\mathbb{Z}_p G_p$. We prove by induction that $[A, {}_iB] = \left(\begin{smallmatrix} 1 & 0 \\ s(g-1)^{i-2} & 1 \end{smallmatrix}\right)$ for $i \geq 2$. We already know that $[A, {}_2B] = \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ s(g-1)^0 & 1 \end{smallmatrix}\right)$. We now assume that $[A, {}_iB] = \left(\begin{smallmatrix} 1 & 0 \\ s(g-1)^{i-2} & 1 \end{smallmatrix}\right)$ and show that $[A, {}_{i+1}B] = \left(\begin{smallmatrix} 1 & 0 \\ s(g-1)^{i-1} & 1 \end{smallmatrix}\right)$. We see that

$$\begin{aligned}[A, {}_{i+1}B] &= [A, {}_iB, B] \\ &= \left[\begin{pmatrix} 1 & 0 \\ s(g-1)^{i-2} & 1 \end{pmatrix}, \begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}\right]\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} [1,g] & 0 \\ -s(g-1)^{i-2}[1,g] - r1^g + s(g-1)^{i-2}g + r & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -s(g-1)^{i-2} - r + s(g-1)^{i-2}g + r & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -s(g-1)^{i-2} + s(g-1)^{i-2}g & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ s(g-1)^{i-1}g & 1 \end{pmatrix}.
\end{aligned} \tag{56}
$$

This calculation completes the induction. Therefore

$$[A,{}_{p+2}B] = \left(\begin{smallmatrix} 1 & 0 \\ s(g-1)^p & 1 \end{smallmatrix}\right). \tag{57}$$

Since $(g-1)^p = g^p - 1$ in $\mathbb{Z}_p G_p$ and G has exponent p, we see that $(g-1)^p = 0$, so

$$[A,{}_{p+2}B] = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \tag{58}$$

and we conclude that M_p is $(p+2)$-Engel.

(5) For each x_i a generator of G_p, we define X_i in M_p to be $\left(\begin{smallmatrix} x_i & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and then let $Y_i = [X_0, X_i]$. Also, let Y be $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We now show by induction on i that $[Y, Y_1, \ldots, Y_i] = \left(\begin{smallmatrix} 1 & 0 \\ u_n & 1 \end{smallmatrix}\right)$, where $u_n = ([x_0,x_1]-1)\cdots([x_0,x_n]-1)$.

We begin by calculating Y_i. Using (55),

$$
\begin{aligned}
Y_i &= [X_0, X_i] \\
&= \left[\begin{pmatrix} x_0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} x_i & 0 \\ 1 & 1 \end{pmatrix}\right] \\
&= \begin{pmatrix} [x_0,x_i] & 0 \\ -[x_0,x_i] - x_0^{x_i} + x_i + 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} [x_0,x_i] & 0 \\ y_i & 1 \end{pmatrix},
\end{aligned} \tag{59}
$$

where $y_i = -[x_0,x_i] - x_0^{x_i} + x_i + 1$.

Now

$$
\begin{aligned}
[Y, Y_1] &= \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} [x_0, x_1] & 0 \\ y_1 & 1 \end{pmatrix}\right] \\
&= \begin{pmatrix} [1, [x_0, x_1]] & 0 \\ -[1, [x_0, x_1]] - y_1 1^{[x_0, x_1]} + [x_0, x_1] + y_1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -1 - y_1 + [x_0, x_1] + y_1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -1 + [x_0, x_1] & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix}.
\end{aligned} \tag{60}
$$

We now assume that $[Y, Y_1, \ldots, Y_i] = \left(\begin{smallmatrix} 1 & 0 \\ u_i & 1 \end{smallmatrix}\right)$ and show that $[Y, Y_1, \ldots, Y_{i+1}] = \left(\begin{smallmatrix} 1 & 0 \\ u_{i+1} & 1 \end{smallmatrix}\right)$. Using (55) once again,

$$
\begin{aligned}
[Y, Y_1, \ldots, Y_{i+1}] &= [[Y, Y_1, \ldots, Y_i], Y_{i+1}] \\
&= \left[\begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}, \begin{pmatrix} [x_0, x_{i+1}] & 0 \\ y_{i+1} & 1 \end{pmatrix}\right] \\
&= \begin{pmatrix} [1, [x_0, x_{i+1}]] & 0 \\ -u_i[1, [x_0, x_{i+1}]] - y_{i+1} 1^{[x_0, x_{i+1}]} + u_i[x_0, x_{i+1}] + y_{i+1} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -u_i - y_{i+1} + u_i[x_0, x_{i+1}] + y_{i+1} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ u_i(-1 + [x_0, x_{i+1}]) & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ u_{i+1} & 1 \end{pmatrix}.
\end{aligned} \tag{61}
$$

This calculation completes the induction. We now consider $u_n = ([x_0, x_1] - 1) \cdots ([x_0, x_n] - 1)$. We can expand this product to see that $u_n = (-1)^n + (-1)^{n-1} S_1 + (-1)^{n-2} S_2 + (-1)^{n-3} S_3 + \cdots - S_{n-1} + S_n$, where

$$
S_1 = \sum_{i=1}^{i=n} [x_0, x_i],
$$

$$
\begin{aligned}
S_2 &= \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n} [x_0, x_{i_1}][x_0, x_{i_2}], \\
S_3 &= \sum_{i_1=1}^{n-2} \sum_{i_2=i_1+1}^{n-1} \sum_{i_3=i_2+1}^{n} [x_0, x_{i_1}][x_0, x_{i_2}][x_0, x_{i_3}], \\
&\dots \\
S_{n-1} &= \sum_{i_1=1}^{2} \cdots \sum_{i_{n-1}=i_{n-1}+1}^{n} [x_0, x_{i_1}] \dots [x_0, x_{i_{n-1}}], \\
S_n &= [x_0, x_1] \dots [x_0, x_n].
\end{aligned}
\tag{62}
$$

Because G_p is defined as the relatively free group on countably many generators $\{x_0, x_1, \dots\}$ with exponent p and nilpotence class 2, the S_j are disjoint sums and S_1 does not equal 0. We conclude that u_n is not zero.

Now, we note that $[Y, Y_1]$ and Y_k for every positive integer k are in $\gamma_1(X_0^{M_p}) = X_0^{M_p}$. If $[Y, Y_1, \dots, Y_{k-1}] \in \gamma_{k-1}(X_0^{M_p})$, then $[Y, Y_1, \dots, Y_k] \in \gamma_k(X_0^{M_p})$. We conclude by induction that $[Y, Y_1, \dots, Y_i] \in \gamma_i(X_0^{M_p})$ for every positive integer i, so $\gamma_i(X_0^{M_p})$ contains a non-trivial element. Thus no term $\gamma_i(X_0^{M_p})$ of the lower central series of $X_0^{M_p}$ is trivial, so $X_0^{M_p}$ is not nilpotent. We conclude that M_p contains an element, X_0, whose normal closure in M_p is not nilpotent, and therefore M_p does not satisfy condition $\mathfrak{L}(\mathfrak{N})$.

$\square$

THEOREM 4.2. *(Gupta and Levin* [**3**]*) For every odd prime* p*, there exists a* p*-group that is* $(p+2)$*-Engel, but that has an element whose normal closure is not nilpotent.*

PROOF. This follows immediately from Lemma 4.1. $\square$

Note that M_3 is a 5-Engel group that does not satisfy condition $\mathfrak{L}(\mathfrak{N})$. However, all 4-Engel groups satisfy condition $\mathfrak{L}(\mathfrak{N}_4)$ [**4**]. Note also that the groups

M_p are not finitely generated. It remains an open question if every finitely generated n-Engel group for n greater than 4 satisfies condition $\mathfrak{L}(\mathfrak{N})$.

Chapter 5

Examples In Which $\mathfrak{E}_n$ Does Not Imply $\mathfrak{L}(\mathfrak{E}_{n-1})$

In Chapter 3, we considered groups that show that condition $\mathfrak{E}_n$ and condition $\mathfrak{L}(\mathfrak{N}_{n-1})$ are not equivalent. In this chapter, we investigate groups that show that $\mathfrak{E}_n$ and $\mathfrak{L}(\mathfrak{E}_{n-1})$ are not equivalent either. We start by discussing an example by Vaughan-Lee [**4**] of a group that is 5-Engel but in which there is an element that has a normal closure that is not 4-Engel. Then we consider an example published by Rips and Shalev in [**6**]. For every odd prime p, they construct a p-group that demonstrates that there exists a natural number n such that an n-Engel p-group does not necessarily satisfy condition $\mathfrak{L}(\mathfrak{E}_{n-1})$. We follow a hint in their paper to construct a 2-group that shows that there exists a natural number n such that a n-Engel 2-group does not necessarily satisfy condition $\mathfrak{L}(\mathfrak{E}_{n-1})$. For each prime p, we then calculate an upper bound on the minimal value of n for which such examples exist.

1. Vaughan-Lee's Example

Vaughan-Lee notes in [**4**] that the largest possible nilpotent 5-Engel group generated by two elements of order 3 contains an element whose normal closure is not 4-Engel. We construct the largest 3-group quotient of Vaughan-Lee's group using GAP and ANUPQ by the following code:

```
RequirePackage("anupq");
f:=FreeGroup(2);
rels:=["f1^3", "f2^3"];
g:=Pq(f: Prime:=3, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
end]);
```

GAP then gives the following information:

```
#I  Class 1 with 2 generators.
#I  Class 2 with 3 generators.
#I  Class 3 with 5 generators.
#I  Class 4 with 7 generators.
#I  Class 5 with 11 generators.
#I  Class 6 with 12 generators.
#I  Class 7 with 14 generators.
#I  Class 8 with 15 generators.
#I  Class 9 with 17 generators.
#I  Class 9 with 17 generators.

gap> n:=LeftNormedComm([g.2,g.1]);
f3
gap> LeftNormedComm([g.1,n,n,n,n]);
f16^2
```

As the GAP output shows, the commutator $[a,[b,a],[b,a],[b,a],[b,a]]$ is not trivial. Because $[b,a]$ is in the normal closure of a, the normal closure of a is not 4-Engel. The group g also can be given by the presentation

$\langle x,y \mid x^3=y^3=[u,{}_5w]=1$, where u and w are any words in x and $y\rangle$. The order of g is 3^{17}. It has pc structure $\{2,3,5,7,11,12,14,15,17\}$. Because the order is the same as that for Vaughan-Lee's group, this group g is the same as his group.

We now consider h, the quotient of g we obtain when we impose the additional relator $[y,x,x,y,x,y,x,y,y]$. This group h is input with the following GAP script:

```
RequirePackage("anupq");
f:=FreeGroup(2);
rels:=[
"f1^3",
"f2^3",
"[f2,f1,f1,f2,f1,f2,f1,f2,f2]"
];


h:=Pq(f: Prime:=3, Relators:=rels,
Identities:=[function(a,b) return LeftNormedComm([a,b,b,b,b,b]);
end]);

n:=LeftNormedComm([h.2,h.1]);

m:=LeftNormedComm([h.1,n,n,n,n]);
```

The group h is 5-Engel and has an element whose normal closure is not 4-Engel. Also, h is of order 3^{16}. The following is a GAP log showing these properties:

```
#I  Class 1 with 2 generators.
#I  Class 2 with 3 generators.
#I  Class 3 with 5 generators.
#I  Class 4 with 7 generators.
#I  Class 5 with 11 generators.
#I  Class 6 with 12 generators.
#I  Class 7 with 14 generators.
#I  Class 8 with 15 generators.
#I  Class 9 with 16 generators.
#I  Class 9 with 16 generators.
gap> m;
f16^2
```

As far as we know, this group h is a new example of a 5-Engel group that contains an element whose normal closure is not 4-Engel.

2. Rips and Shalev Counterexample

The above example of Vaughan-Lee and the group h from the previous section show that condition $\mathfrak{E}_5$ does not imply condition $\mathfrak{L}(\mathfrak{E}_4)$. However, they are 3-groups, and leave open the question of how these conditions relate for groups without elements of order 3. In this section we explain the details of a family of examples by Rips and Shalev [**6**]. For each odd prime p, a group from this family shows that there must exist some positive integer n_p such that an n_p-Engel p-group exists that has an element whose normal closure is not $(n_p - 1)$-Engel. We also construct a similar 2-group. The 2-group construction follows a hint in the paper of Rips and Shalev, and our examination of the construction for odd primes provides many more details than their paper does.

2.1. Definitions. We need a few definitions before we can construct the examples of Rips and Shalev.

DEFINITION 5.1. Let F_t be a free group of rank t, with $t > 1$. Then a *sequence of basic commutators* in F_t is an infinite sequence $c_1, c_2, \dots$ of elements of F_t such that:

(1) each c_i has associated with it a positive integer w_i called its weight;
(2) the c_i are ordered by weight, meaning that if $j > i$, then $w_j \geq w_i$;
(3) the basic commutators of weight 1, namely $c_1, \dots, c_t$, are the free generators of F_t arranged in some order;
(4) if $w_k > 1$, then c_k is described explicitly by $[c_j, c_i]$, where $j > i$ and $w_j + w_i = w_k$. Also, if $w_j > 1$, then there exist $q < j$ and $r \leq i$ such that $c_j = [c_q, c_r]$.

A *basic commutator* is any c_i in some sequence of basic commutators for F_t for some integer $t > 1$.

Any finitely generated free group has many different sequences of basic commutators, based on choices of the order of basic commutators of a given weight. A choice made at one weight will affect what choices are available at later weights.

We will often use basic commutators in the relatively free group of exponent e, nilpotence class c, and rank d. We will refer to this relatively free group as $H_{d,e,c}$. A sequence of basic commutators for $H_{d,e,c}$ is derived from a sequence of basic commutators of F_d. It contains images under the natural map of all the basic commutators of F_d of weight at most c. By Theorem 11.2.4 of [**63**], if $\{c_1, \dots, c_s\}$ are the elements in an ordered sequence of basic commutators for F_d that are of weight c or less, then an arbitrary element f of F_d has a unique representation relative to this sequence of basic commutators:

$$f = c_1^{e_1} \cdots c_s^{e_s} \bmod \gamma_{s+1}(F_d), \tag{63}$$

where e_i is a non-negative integer. In $H_{d,e,c}$, each element h has a representation (not necessarily unique) relative to a particular sequence of basic commutators:

$$h = \hat{c}_1^{e_1} \cdots \hat{c}_s^{e_s}, \tag{64}$$

where $\hat{c}_i$ is the image of c_i under the natural quotient map and each e_i is an integer mod e. It should be noted that the sequence of basic commutators for $H_{d,e,c}$ is finite.

From this discussion, it follows that:

LEMMA 5.1. *Let s be the number of elements of a sequence of basic commutators in $H_{d,e,c}$. Then the order of $H_{d,e,c}$ is at most e^s.*

DEFINITION 5.2. A group G is a *Baer group* if every cyclic subgroup of G is subnormal in G.

DEFINITION 5.3. A group G is *p-abelian* if its commutator subgroup G' is a finite p-group.

It should be noted that some authors use the term p-abelian differently. We will find FC-elements and the FC center of a group to be helpful.

DEFINITION 5.4. An element a of group G is called an *FC-element* if the set $\{a^g \mid g \in G\}$ of all conjugates of a in G is finite.

DEFINITION 5.5. The *FC-center* of group G is the set of all FC-elements of G.

The Rips and Shalev examples make use of several ways of moving between rings, algebras, and groups.

DEFINITION 5.6. Let R be a ring. A *unit* of R is an element $r \in R$ that possesses a multiplicative inverse in R. We use the notation $U(R)$ for the multiplicative group of units of R.

Definition 5.7. Let G be a group and F a field. The *group algebra* FG is defined to be the set of formal sums $\sum_{g\in G} r_g g$, where $r_g \in F$ and all but finitely many $r_g = 0$. Addition is defined on FG by

$$(\sum_{g\in G} r_g g) + (\sum_{g\in G} r'_g g) = \sum_{g\in G}(r_g + r'_g)g, \tag{65}$$

and multiplication is defined on FG by

$$(\sum_{g\in G} r_g g)(\sum_{g\in G} r'_g g) = \sum_{g\in G}(\sum_{xy=g} r_x r'_y)g. \tag{66}$$

Furthermore, scalar multiplication of F on FG is given by

$$r(\sum_{g\in G} r_g g) = \sum_{g\in G}(r r_g)g. \tag{67}$$

Definition 5.8. Let G be a group and F a field. The *augmentation ideal* of the group ring FG is the subset of FG of those elements $\sum_{g\in G} r_g g$ for which $\sum r_g = 0$.

We also need to define a few more concepts related to Lie rings.

Definition 5.9. Let R be a Lie ring and let x be an element of R. Then the mapping $ad_x : R \to R$ is given by $ad_x(a) = (x, a)$ for each a in R. This map is called the *inner derivation* of the Lie ring R at the element x.

Definition 5.10. Let R be a Lie ring and let $x \in R$. An *n-stable series for ad_x* is a decreasing sequence of Lie subrings

$$R = R_0 \supset R_1 \supset \cdots \supset R_n = 0, \tag{68}$$

where $(R_{i+1}, R_i) \subset R_{i+1}$ and where $(x, R_i) \subset R_{i+1}$. The element x of a Lie ring R is *n-Baer* if R has a n-stable series for ad_x, and the Lie ring is *Baer* if every element is n-Baer for some n. If there is a positive integer n such that R has an n-stable series for ad_x for every element x of R, then R is *n-Baer*. A Lie ring is called *Baer-bounded* if it is n-Baer for some positive integer n.

We also use a concept from number theory in our calculations:

DEFINITION 5.11. Let i be a positive integer, with the factorization $p_1^{e_1} \dots p_s^{e_s}$, where $p_1, \dots, p_s$ are distinct primes. Then we define the *Moebius function* μ by

$$\mu(i) = \begin{cases} 1 & \text{for } i = 1, \\ (-1)^s & \text{if } e_k = 1 \text{ for all } k \text{ in } \{1, 2, \dots, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

2.2. General Lemmas. We start by gathering several results about semidirect products. Let C and D be groups. First, observe that the inverse of the element (c, d) in the semidirect product $C \rtimes D$ is $(c^{-d^{-1}}, d^{-1})$.

LEMMA 5.2. *In the semidirect product $C \rtimes D$, if $m, n \in C$ and $s, t \in D$, then*

$$(m, s)^{(n,t)} = ((n^{-1}mn^s)^{t^{-1}}, t^{-1}st) \tag{69}$$

and

$$[(m, s), (n, t)] = (m^{-s^{-1}}(n^{-1}mn^s)^{t^{-1}s^{-1}}, s^{-1}t^{-1}st). \tag{70}$$

If D is an abelian group, then

$$(m, s)^{(n,t)} = ((n^{-1}mn^s)^{t^{-1}}, s) \tag{71}$$

and

$$[(m, s), (n, t)] = (m^{-s^{-1}}n^{-t^{-1}s^{-1}}m^{s^{-1}t^{-1}}n^{t^{-1}}, 1). \tag{72}$$

If C and D are both abelian, then

$$[(m, s), (n, t)] = ((m^{-1}m^{t^{-1}})^{s^{-1}}(n^{-s^{-1}}n)^{t^{-1}}, 1). \tag{73}$$

PROOF. Let $m, n \in C$ and $s, t \in D$, so (m, s) and (n, t) are in $C \rtimes D$. Then

$$
\begin{aligned}
(m,s)^{(n,t)} &= (n,t)^{-1}(m,s)(n,t) \\
&= (n^{-t^{-1}}, t^{-1})(mn^s, st) \\
&= (n^{-t^{-1}}(mn^s)^{t^{-1}}, t^{-1}st) \\
&= ((n^{-1}mn^s)^{t^{-1}}, t^{-1}st).
\end{aligned} \tag{74}
$$

Also,

$$
\begin{aligned}
[(m,s),(n,t)] &= (m,s)^{-1}(m,s)^{(n,t)} \\
&= (m^{-s^{-1}}, s^{-1})((n^{-1}mn^s)^{t^{-1}}, t^{-1}st) \\
&= (m^{-s^{-1}}(n^{-1}mn^s)^{t^{-1}s^{-1}}, s^{-1}t^{-1}st).
\end{aligned} \tag{75}
$$

The abelian cases now follow easily. □

We also need a formula for exponents in a semidirect product.

Lemma 5.3. *For any positive integer k, if (m,s) is an element of the semidirect product $C \rtimes D$, then $(m,s)^k = (mm^s \dots m^{s^{k-1}}, s^k)$.*

Proof. The proof follows by induction on k. We first notice that the base case of the induction is trivial. Now

$$
\begin{aligned}
(m,s)^k &= (m,s)^{k-1}(m,s) \\
&= (mm^s \dots m^{s^{k-2}}, s^{k-1})(m,s) \\
&= (mm^s \dots m^{s^{k-2}} m^{s^{k-1}}, s^{k-1}s) \\
&= (mm^s \dots m^{s^{k-1}}, s^k).
\end{aligned} \tag{76}
$$

□

We also use a basic result about solvability of quotients of Lie rings.

Lemma 5.4. *Let L be a Lie ring and let M be an ideal of L. If M is Lie-solvable and L/M is commutative, then L is also Lie-solvable.*

PROOF. Because L/M is commutative, $L^{(1)} = [L, L]$ is a subset of M. Since M is solvable, L is also solvable. □

We will use the following properties:

DEFINITION 5.12. We say that a group G has property $\mathfrak{C}$ if there exists a sequence $\{\beta_1, \beta_2, \dots\}$ of positive integers such that for every positive integer d, every subgroup H of G with d generators is nilpotent of class at most β_d.

DEFINITION 5.13. A group G has property $\mathfrak{D}$ if there is a sequence $\{\gamma_1, \gamma_2, \dots\}$ of positive integers such that for every positive integer d, for every subgroup H of G with d generators, the order of H is at most γ_d.

We will use the following results about p-groups:

LEMMA 5.5. *Let G be a finite p-group. If G has order v, then G is nilpotent of class at most $log_p(v) - 1$.*

PROOF. This follows by result 5.3.1 (i) in [**64**], which states that if G is a group of order p^{m+1}, then G is nilpotent of class at most m. □

LEMMA 5.6. *The number of elements in a set of basic commutators of $H_{d,e,n}$ is at most*

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{\lfloor n/i \rfloor} \left(\frac{\mu(i)}{i} \right) \cdot \frac{d^j}{j} \right). \tag{77}$$

PROOF. Witt's theorem ([**63**], page 169) states that the number of basic commutators of weight m for a free group on d generators is given by

$$\sum_{i \mid m} \left(\frac{\mu(i) d^{m/i}}{m} \right). \tag{78}$$

Because the terms of a sequence of basic commutators in $H_{d,e,n}$ are images of the terms of a sequence of basic commutators of weight at most n in the free group on d

generators, the number of basic commutators of weight m, where $1 \leq m \leq n$, in a sequence for $H_{d,e,n}$ is at most

$$\sum_{i \mid m} \left(\frac{\mu(i) d^{m/i}}{m} \right). \tag{79}$$

The total number of basic commutators is the sum of the number of basic commutators of each weight up to the nilpotence class. Thus, the number of basic commutators for $H_{d,e,n}$ of all weights is at most

$$\sum_{m=1}^{n} \left(\sum_{i \mid m} \left(\frac{\mu(i) d^{m/i}}{m} \right) \right) = \sum_{i=1}^{n} \left(\sum_{m \leq n, i \mid m} \left(\frac{\mu(i) d^{m/i}}{m} \right) \right) \tag{80}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{\lfloor n/i \rfloor} \left(\frac{\mu(i) d^{j}}{ji} \right) \right), \tag{81}$$

where $j = m/i$. □

Proposition 5.7. *Let G be a d-generator group with exponent e and nilpotence class at most c. Then G has order less than or equal to:*

$$e^{\left(\sum_{i=1}^{c} \left(\sum_{j=1}^{\lfloor c/i \rfloor} \left(\frac{\mu(j)}{j} \right) \cdot \frac{d^i}{i} \right) \right)}. \tag{82}$$

Proof. We first note that G is a quotient of $H_{d,e,c}$. If b is the number of basic commutators in a sequence of basic commutators for $H_{d,e,c}$, then, by Lemma 5.1, the order of $H_{d,e,c}$ is at most e^b. Hence, by Lemma 5.6, $H_{d,e,c}$ has order at most

$$e^{\left(\sum_{i=1}^{c} \left(\sum_{j=1}^{\lfloor c/i \rfloor} \left(\frac{\mu(j)}{j} \right) \cdot \frac{d^i}{i} \right) \right)}. \tag{83}$$

□

We may now prove the following Corollary:

Corollary 5.8. *Let G be a p-group with exponent e and property $\mathfrak{C}$. Then G has property $\mathfrak{D}$.*

Proof. Because G satisfies property $\mathfrak{C}$, there is a sequence $\{\beta_1, \beta_2, \dots\}$ of positive integers such that for every positive integer d, every subgroup H of G with d generators is nilpotent of class at most β_d. We need to show that there is a sequence $\{\gamma_1, \gamma_2, \dots\}$ such that the order of a d-generator subgroup of G is at most γ_d. Let H be a d-generator subgroup of G. Since G has property $\mathfrak{C}$, we know that H has nilpotence class at most β_d. By hypothesis, H is a p-group of exponent e. By Proposition 5.7, H is of order at most

$$e^{\left(\sum_{i=1}^{\beta_d}\left(\sum_{j=1}^{\lfloor \beta_d/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{d^i}{i}\right)\right)}, \tag{84}$$

so we may set

$$\gamma_d = e^{\left(\sum_{i=1}^{\beta_d}\left(\sum_{j=1}^{\lfloor \beta_d/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{d^i}{i}\right)\right)}. \tag{85}$$

□

Proposition 5.9. *Let G be a p-group with exponent e and property $\mathfrak{C}$, with associated sequence $\{\beta_d\}$ and let s be a positive integer. Let H be the wreath product $H = G \wr \mathbb{Z}_p$ and let K be an s-generator subgroup of H. Then K has order less than or equal to:*

$$p \cdot e^{p\left(\sum_{i=1}^{\beta_{sp}}\left(\sum_{j=1}^{\lfloor \beta_{sp}/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(sp)^i}{i}\right)\right)}. \tag{86}$$

Proof. Let $\{x_1, \dots, x_s\}$ be a set of generators of K, and let B be the base group of the wreath product $H = G \wr \mathbb{Z}_p$. Because $G \wr \mathbb{Z}_p = B \rtimes \mathbb{Z}_p$, we can write each x_i as (b_i, z_i), where b_i is an element of B and z_i is an element of $\mathbb{Z}_p$. Since B is the direct product of p copies of G, each b_i can be written $(g_{i,1}, \dots, g_{i,p})$, where $g_{i,j}$ is an element of G. Let $\psi_j : B \to G$ be the mapping determined by $\psi_j((g_1, \dots, g_p)) = g_j$. Now define $L = \langle \psi_j(g_{i,j}) \mid i \in \{1, \dots, s\}, j \in \{1, \dots, p\}\rangle$. In other words, L is the subgroup generated by the elements of G that are used to construct $\{x_1, \dots, x_s\}$. Now let $M = L \wr \mathbb{Z}_p$. We note that L is a subgroup of G, so

M is a subgroup of H. Also, K is a subgroup of M. The group L is a subgroup of G that has at most sp generators, and so by Proposition 5.7, the order of L is at most

$$e^{\left(\sum_{i=1}^{\beta_{sp}}\left(\sum_{j=1}^{\lfloor \beta_{sp}/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(sp)^i}{i}\right)\right)}. \tag{87}$$

Thus the base group of M has order at most

$$e^{\left(\sum_{i=1}^{\beta_{sp}}\left(\sum_{j=1}^{\lfloor \beta_{sp}/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(sp)^i}{i}\right)\right)p}, \tag{88}$$

and hence M has order at most

$$p \cdot e^{p\left(\sum_{i=1}^{\beta_{sp}}\left(\sum_{j=1}^{\lfloor \beta_{sp}/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(sp)^i}{i}\right)\right)}. \tag{89}$$

Because K is a subgroup of M, the order of K is also at most

$$p \cdot e^{p\left(\sum_{i=1}^{\beta_{sp}}\left(\sum_{j=1}^{\lfloor \beta_{sp}/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(sp)^i}{i}\right)\right)}. \tag{90}$$

$\square$

Corollary 5.10 follows immediately from Proposition 5.9.

COROLLARY 5.10. *Let G be a p-group of exponent e that has property $\mathfrak{C}$. Let H be the wreath product $H = G \wr \mathbb{Z}_p$. Then H has property $\mathfrak{D}$.*

PROPOSITION 5.11. *Let G be a p-group of exponent e that has property $\mathfrak{C}$, with associated sequence $\{\beta_d\}$. Let H be the wreath product $H = G \wr \mathbb{Z}_p$, and let K be an s-generator subgroup of H. Then K is nilpotent of class at most:*

$$p \cdot log_p e \cdot \left(\sum_{i=1}^{\beta_{sp}}\left(\sum_{j=1}^{\lfloor \beta_{sp})/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(sp)^i}{i}\right)\right). \tag{91}$$

PROOF. In Proposition 5.9, we showed that the order of K is at most

$$p \cdot e^{p\left(\sum_{i=1}^{\beta_{sp}}\left(\sum_{j=1}^{\lfloor \beta_{sp}/i \rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(sp)^i}{i}\right)\right)}. \tag{92}$$

By Lemma 5.5, the nilpotence class of K is at most

$$p \cdot log_p e \cdot \left(\sum_{i=1}^{\beta_{sp}} \left(\sum_{j=1}^{\lfloor \beta_{sp}/i \rfloor} \left(\frac{\mu(j)}{j} \right) \cdot \frac{(sp)^i}{i} \right) \right). \tag{93}$$

□

The final three introductory results we use are quoted without proof.

LEMMA 5.12. *(Passi, Passman, and Seghal* [**65**]*) Let p be a prime number and let G be a group. If K is a field of characteristic p, then*

(a) the group ring KG is Lie-nilpotent if and only if G is p-abelian and nilpotent;

(b) if $p \neq 2$, then KG is Lie-solvable if and only if G is p-abelian;

(c) if $p = 2$, then KG is Lie-solvable if and only if G has a 2-abelian subgroup of index at most 2.

LEMMA 5.13. *(Bovdi and Kripta (or Hripta)* [**66**]*, translated* [**67**]*) Let p be a prime number. Let K be a field of characteristic $p \geq 3$. Let G be a periodic group and let P be a Sylow p-subgroup of G. Then the group of units U(KG) of the group ring KG is solvable if and only if either*

(1) *the commutator subgroup of G is a finite p-group, or*
(2) *K is a field of 3 elements, P is finite and normal in G, and one of the following conditions holds for $E = G/P$:*
    (a) *E has an abelian normal subgroup A of exponent 8, where $D = E/A$ has order 2 and $dad^{-1} = a^3$ for all $d \in D$ and $a \in A$.*
    (b) *E is a 2-group of nilpotency class 2, and the nonabelian subgroups of its homomorphic images contain no central elements of order 4.*

LEMMA 5.14. *(Bovdi and Kripta (or Hripta)* [**66**]*, translated* [**67**]*) Let K be a field of characteristic 2, and let G be a periodic group. Then the group U(KG) is* *solvable if and only if G contains a finite normal 2-subgroup N such that $G_1 = G/N$ satisfies one of the following conditions:*

(1) *G_1 is abelian.*

(2) *G_1 is the direct product of an abelian group W having no elements of order 2 and a 2-group B such that B contains an abelian normal subgroup of index 2, and the quotient group of B by its center has bounded exponent.*

(3) *K is the field of 2 elements and the group G_1 is an extension of the direct product $M \times W$ of an abelian 2-group M of bounded exponent and an elementary abelian 3-group W by the group $\langle b \rangle$ of order 2, with, moreover, $bwb^{-1} = w^{-1}$ for all $w \in W$.*

2.3. Construction for an Odd Prime. In this subsection we discuss (and give many more details than the published account of) a proof of the theorem:

THEOREM 5.15. *(Rips and Shalev [**6**]) For every prime $p \neq 2$, there is some positive integer n for which there is an n-Engel p-group that has an element whose normal closure in the whole group is not $(n-1)$-Engel.*

We prove Theorem 5.15 by constructing a sequence of groups and rings G_p, R_p, H_p, and M_p. We use the group M_p to prove by contradiction that for p-groups, condition $\mathfrak{E}_n$ does not imply condition $\mathfrak{L}(\mathfrak{E}_{n-1})$ for some positive integer n.

CONSTRUCTION 2 (Construction of G_p). *(Rips and Shalev [**6**]) For each prime $p \neq 2$, let*
$I_p = \langle (a_i, z_i) \mid a_i^p = z_i^p = [a_i, a_j] = [a_i, z_j] = [z_i, z_j] = 1$ for $i, j \in \{1, 2, \dots\} \rangle$, so that I_p is $(C_p \times C_p)^\omega$. Let X_p be the group $\langle x \mid x^p = 1 \rangle$ and let G_p be the group $I_p \rtimes X_p$, where the action of X_p on I_p is defined by $a_i^x = a_i z_i$ and $z_i^x = z_i$.

LEMMA 5.16. *(Rips and Shalev, [**6**]) The following hold:*

(1) *G_p is nilpotent of class 2.*

(2) *G_p is of exponent p.*

(3) G_p *has an abelian subgroup of index* p.

(4) G_p *is not* p*-abelian.*

(5) $\mathbb{Z}_p G_p$ *is not Lie-solvable.*

PROOF. (1) Because X_p and I_p are abelian, for $m, n \in I_p$ and $s, t \in X_p$, by Lemma 5.2,

$$[(m, s), (n, t)] = ((m^{-1} m^{t^{-1}})^{s^{-1}} (n^{-s^{-1}} n)^{t^{-1}}, 1). \tag{94}$$

Now if $m = (a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)$ and $n = (a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)$, and if also $s = x^i$ and $t = x^j$, where w_k, v_k, $\hat{w}_k$, $\hat{v}_k$, i, and j are all integers,then

$$\begin{aligned}
[(m, s), (n, t)] &= ((m^{-1} m^{t^{-1}})^{s^{-1}} (n^{-s^{-1}} n)^{t^{-1}}, 1) \\
&= (((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)^{-1} (a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)^{x^{-j}})^{x^{-i}} \\
&\quad ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-x^{-i}} (a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots))^{x^{-j}}, 1) \\
&= (((a_1^{-v_1} z_1^{-w_1} a_2^{-v_2} z_2^{-w_2} \dots)(a_1^{x^{-j} v_1} z_1^{x^{-j} w_1} a_2^{x^{-j} v_2} z_2^{x^{-j} w_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-x^{-i} \hat{v}_1} z_1^{-x^{-i} \hat{w}_1} a_2^{-x^{-i} v_2} z_2^{-x^{-i} \hat{w}_2} \dots)(a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots))^{x^{-j}}, 1) \\
&= (((a_1^{-v_1} z_1^{-w_1} a_2^{-v_2} z_2^{-w_2} \dots)(a_1^{x^{-j} v_1} z_1^{w_1} a_2^{x^{-j} v_2} z_2^{w_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-x^{-i} \hat{v}_1} z_1^{-\hat{w}_1} a_2^{-x^{-i} \hat{v}_2} z_2^{-\hat{w}_2} \dots)(a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots))^{x^{-j}}, 1) \qquad (95) \\
&= (((a_1^{-v_1} a_2^{-v_2} \dots)(a_1^{x^{-j} v_1} a_2^{x^{-j} v_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-x^{-i} \hat{v}_1} a_2^{-x^{-i} \hat{v}_2} \dots)(a_1^{\hat{v}_1} a_2^{\hat{v}_2} \dots))^{x^{-j}}, 1) \\
&= (((a_1^{-v_1} a_2^{-v_2} \dots)(a_1^{v_1} z_1^{-j v_1} a_2^{v_1} z_2^{-j v_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-\hat{v}_1} z_1^{i \hat{v}_1} a_2^{-\hat{v}_2} z_2^{i \hat{v}_2} \dots)(a_1^{\hat{v}_1} a_2^{\hat{v}_2} \dots))^{x^{-j}}, 1) \\
&= ((z_1^{-j v_1} z_2^{-j v_2} \dots)^{x^{-i}} (z_1^{i \hat{v}_1} z_2^{i \hat{v}_2} \dots)^{x^{-j}}, 1) \\
&= ((z_1^{-j v_1} z_2^{-j v_2} \dots)(z_1^{i \hat{v}_1} z_2^{i \hat{v}_2} \dots), 1) \\
&= ((z_1^{i \hat{v}_1 - j v_1} z_2^{i \hat{v}_2 - j v_2} \dots), 1).
\end{aligned}$$

We have shown that any commutator of elements in G_p is contained in the subgroup $\langle z_k \mid k \in \{1, 2, \dots\} \rangle$, and so is in $Z(G_p)$. Hence G_p is nilpotent of class 2.

(2) We show that G_p is of exponent p by computing the pth power of an arbitrary element $((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots), x^j)$ of G_p, where w_k, v_k, and j are all integers. By Lemma 5.3 and the action of X_p on I_p,

$$\begin{aligned}
((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots), x^j)^p &= ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)(a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)^{x^j} \\
&\quad \cdots (a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)^{(x^j)^{p-1}}, (x^j)^p) \\
&= ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)(a_1^{v_1} z_1^{w_1+jv_1} a_2^{v_2} z_2^{w_2+jv_2} \dots) \\
&\quad \cdots (a_1^{v_1} z_1^{w_1+(p-1)jv_1)} a_2^{v_2} z_2^{w_2+(p-1)jv_2} \dots), (x^p)^j) \\
&= ((a_1^{v_1 p} z_1^{pw_1+j(1+\cdots+(p-1))v_1} a_2^{pv_2} \\
&\quad z_2^{pw_2+j(1+\cdots+(p-1))v_2} \dots), (x^p)^j) \\
&= (((a_1^p)^{v_1} z_1^{pw_1+v_1 jp(p-1)/2} a_2^p z_2^{pw_2+v_2 jp(p-1)/2)} \dots), (x^p)^j).
\end{aligned} \tag{96}$$

Because p is an odd prime and the a_i, the z_i, and x are all of order p, we conclude that G_p is of exponent p.

(3) $((C_p \times C_p)^\omega, 1)$ is an abelian subgroup of G_p of index p.

(4) In part (1) of this proof, we show that the subgroup $\langle z_i \mid i \in \{1, 2, \dots\}\rangle$ contains G_p'. Each z_i is in G_p' because $[a_i, x] = z_i$. Hence G_p' is the subgroup $\langle z_i \rangle$, and so is infinite. Hence G_p is not p-abelian.

(5) This follows from (4) and part (b) of Lemma 5.12.

□

Construction 3 (Construction of R_p). *(Rips and Shalev, [**6**]) For each prime $p \neq 2$, define R_p to be the ring*

$$\mathbb{Z}_p G_p \Delta F_p, \tag{97}$$

where $F_p = FC(G_p)$, the subgroup of all FC-elements of G_p, and ΔF_p is the augmentation ideal of $\mathbb{Z}_p F_p$.

We will also need the notation (used in [**68**]) that $\Delta(G_p, F_p)$ is the left ideal of $\mathbb{Z}_p G_p$ generated by $\{h - 1 \mid h \in F_p\}$. It should be noted that some authors use Δ differently.

LEMMA 5.17. *(Rips and Shalev, [**6**]) The following claims hold:*

(1) $F_p = (I_p, 1)$, *which is an elementary abelian* p*-group.*

(2) $G_p \cong F_p \rtimes C_p$.

(3) $a^p = 0$ *for every* $a \in \Delta(F_p)$.

(4) $R_p = \Delta(G_p, F_p)$.

(5) $R_p = \bigoplus_{0 \le i < p} x^i \cdot \Delta F_p$.

PROOF. (1) We first compute the conjugate of one arbitrary element of G_p by another. Let $g = ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots), x^s)$ and $\hat{g} = ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots), x^t)$ be elements of G_p, where the w_i, v_i, $\hat{w}_i$, $\hat{v}_i$, s, and t are integers modulo p. Then, by the observation earlier that the inverse of the element (c, d) in the semidirect product $C \rtimes D$ is $(c^{-d^{-1}}, d^{-1})$ and by the action of X_p on I_p,

$$\begin{aligned}
g^{\hat{g}} &= ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots), x^s)^{((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots), x^t)} \\
&= ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots), x^t)^{-1} \\
&\quad \cdot ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots), x^s)((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots), x^t) \\
&= ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-x^{-t}}, x^{-t}) \\
&\quad \cdot ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)(a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{x^s}, x^{s+t}) \\
&= ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-x^{-t}}, x^{-t}) \\
&\quad \cdot ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)(a_1^{x^s \hat{v}_1} z_1^{x^s \hat{w}_1} a_2^{x^s \hat{v}_2} z_2^{x^s \hat{w}_2} \dots), x^{s+t}) \\
&= ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-x^{-t}}, x^{-t}) \\
&\quad \cdot ((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)(a_1^{\hat{v}_1} z_1^{s\hat{v}_1 + \hat{w}_1} a_2^{\hat{v}_2} z_2^{s\hat{v}_2 + \hat{w}_2} \dots), x^{s+t}) \qquad (98) \\
&= ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-x^{-t}}, x^{-t}) \\
&\quad \cdot (a_1^{v_1 + \hat{v}_1} z_1^{w_1 + s\hat{v}_1 + \hat{w}_1} a_2^{v_2 + \hat{v}_2} z_2^{w_2 + s\hat{v}_2 + \hat{w}_2} \dots), x^{s+t}) \\
&= ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-x^{-t}} \\
&\quad \cdot (a_1^{v_1 + \hat{v}_1} z_1^{w_1 + s\hat{v}_1 + \hat{w}_1} a_2^{v_2 + \hat{v}_2} z_2^{w_2 + s\hat{v}_2 + \hat{w}_2} \dots)^{x^{-t}}, x^{-t+s+t}) \\
&= ((a_1^{-\hat{v}_1} z_1^{-\hat{w}_1} a_2^{-\hat{v}_2} z_2^{-\hat{w}_2} \dots) \\
&\quad \cdot (a_1^{v_1 + \hat{v}_1} z_1^{w_1 + s\hat{v}_1 + \hat{w}_1} a_2^{v_2 + \hat{v}_2} z_2^{w_2 + s\hat{v}_2 + \hat{w}_2} \dots))^{x^{-t}}, x^{-t+s+t}) \\
&= ((a_1^{v_1} z_1^{w_1 + s\hat{v}_1} a_2^{v_2} z_2^{w_2 + s\hat{v}_2} \dots))^{x^{-t}}, x^s) \\
&= (a_1^{v_1} z_1^{w_1 + s\hat{v}_1 - tv_1} a_2^{v_2} z_2^{w_2 + s\hat{v}_2 - tv_2} \dots, x^s).
\end{aligned}$$

If $s = 0$, then there are p choices for t, so g has p conjugates, but if s is not 0, then g has infinitely many conjugates because there are infinitely many choices for the sequence $\{\hat{v}_i\}$. We conclude that $F_p = (I_p, 1)$.

(2) We defined G_p to be $I_p \rtimes X_p$. Since X_p is $\mathbb{Z}_p$ and we have shown that $I_p \cong (I_p, 1) = F_p$, we conclude that $G_p \cong F_p \rtimes \mathbb{Z}_p$.

(3) By the definition of augmentation ideal, any element a of ΔF_p is of the form $r_1 g_1 + r_2 g_2 + \cdots + r_m g_m$, where $r_i \in \mathbb{Z}_p$, $g_i \in G_p$, and $\sum r_i = 0$. In a commutative ring over $\mathbb{Z}_p$, we know $(i+j)^p = i^p + j^p$, so

$$\begin{aligned} a^p &= (r_1 g_1 + r_2 g_2 + \cdots + r_{m-1} g_{m-1} + r_m g_m)^p \\ &= (r_1 g_1)^p + (r_2 g_2)^p + \cdots + (r_{m-1} g_{m-1})^p + (r_m g_m)^p \\ &= r_1^p 1_G + r_2^p 1_G + \cdots + r_{m-1}^p 1_G + r_m^p 1_G. \end{aligned} \tag{99}$$

In $\mathbb{Z}_p$, we know $r_i^p = r_i$, so

$$\begin{aligned} a^p &= r_1 1_G + r_2 1_G + \cdots + r_{m-1} 1_G + r_m 1_G \\ &= (r_1 + r_2 + \cdots + r_{m-1} + r_m) 1_G \\ &= 0 \cdot 1_G \\ &= 0. \end{aligned} \tag{100}$$

(4) By part (i) of exercise 2 on page 138 of [**68**], $R_p = \Delta(G_p, F_p)$.

(5) We first note that $\{x^i \mid i \in \{0, \ldots, p-1\}\}$ is a transversal of F_p in G_p. By Proposition 3.3.3 of [**68**], $\{x^i(h-1) \mid i \in \{0, \ldots, p-1\}, h \in F_p, h \neq 1\}$ is a basis for $\Delta(G_p, F_p)$. However, $\Delta(G_p, F_p)$ is defined as the left ideal of $\mathbb{Z}_p G_p$ generated by $\{h - 1 \mid h \in F_p\}$. This gives the desired conclusion.

□

For u a unit of $\mathbb{Z}_p G_p$ and $r \in \mathbb{Z}_p G_p$, we can use the notation r^u to denote $u^{-1} r u$ in $\mathbb{Z}_p G_p$. In particular, any element of G_p is a unit in $\mathbb{Z}_p G_p$. By part (5) of Lemma

5.17, every y in R_p may be written uniquely in the form $\sum_{0\leq i<p} x^i \cdot y_i$, where $y_i \in \Delta F_p$. Let $S_y = \{y_i^{x^j} : 0 \leq i, j < p\}$. Also, let $n_p = 1 + (p-1)p^2$.

LEMMA 5.18. *(Rips and Shalev* [**6**]*) Let y be an element of R_p. Then the following statements hold:*

(1) *$b^p = 0$ for every element $b \in S_y$.*
(2) *$\mid S_y \mid \leq p^2$.*
(3) *$S_y^{n_p} = 0$.*
(4) *$S_y \cdot \mathbb{Z}_p G_p = \mathbb{Z}_p G_p \cdot S_y$.*
(5) *$(S_y \cdot \mathbb{Z}_p G_p)^{n_p} = (S_y)^{n_p} \cdot \mathbb{Z}_p G_p = 0$.*
(6) *$S_y \cdot \mathbb{Z}_p G_p$ is a 2-sided ideal of $\mathbb{Z}_p G_p$.*
(7) *$y \in S_y \cdot \mathbb{Z}_p G_p$.*
(8) *$(\mathbb{Z}_p G_p y \mathbb{Z}_p G_p)^{n_p} = 0$.*
(9) *$r^{n_p} = 0$ for every r in R_p.*
(10) *R_p is Baer-bounded.*
(11) *$\mathbb{Z}_p G_p / R_p \cong \mathbb{Z}_p C_p$ as rings, where C_p is $\langle x \rangle$.*
(12) *$\mathbb{Z}_p G_p / R_p$ is a commutative ring.*
(13) *R_p is not a Lie-solvable ring.*

PROOF. (1) Any $b \in S_y$ is of the form $y_i^{x^j}$, where y_i is in $\Delta(F_p)$, so $b^p = (y_i^{x^j})^p = (y_i^p)^{x^j} = 0$ by part (4) of Lemma 5.17.

(2) $\mid S_y \mid \leq p^2$ because the elements of S_y are determined by making two choices, each of which has p possibilities.

(3) We want to show that any product of n_p elements of S_y is trivial. We first note that F_p is abelian, and that any conjugate of an element of F_p by x remains in F_p. Thus the elements of S_y commute. Because $b^p = 0$ for each $b \in S_y$, a product that contains any factor p or more times is trivial. Since S_y has at most p^2 elements, $p^2(p-1)$ is the largest number of factors in a nontrivial product.

(4) We want to show that $S_y \cdot \mathbb{Z}_pG_p = \mathbb{Z}_pG_p \cdot S_y$. To do so, for s an element of S_y, we show that each element of $s\mathbb{Z}_pG_p$ is a sum of elements of the form t_kg_kz, where $t_k \in \mathbb{Z}_p$, $g_k \in G_p$, and $z \in S_y$. Let $v \in s\mathbb{Z}_pG_p$, so $v = s\sum t_kg_k$ where $t_k \in \mathbb{Z}_p$ and $g_k \in G_p$. Then
$v = s\sum t_kg_k = \sum(st_kg_k) = \sum t_k(sg_k) = \sum t_kg_ks^{g_k}$. We now consider one of the summands $t_kg_ks^{g_k}$. We need to show that s^{g_k} is in S_y. Because s is in S_y, we can write $s = y_i^{x^j}$, where $y_i \in \Delta(F_p)$. Thus $s^{g_k} = y_i^{x^jg_k}$. We know that $x^jg_k \in G_p$, so $x^jg_k = fx^m$, where $f \in F_p$ and $m \in \{0, \ldots, p-1\}$. Then $s^{g_k} = y_i^{fx^m} = (y_i^f)^{x^m}$. Now we consider y_i^f. Because $y_i \in \Delta(F_p)$, it is of the form $r_1g_1 + r_2g_2 + \cdots + r_mg_m$, where $r_i \in \mathbb{Z}_p$, $g_i \in F_p$, and $\sum r_i = 0$. We calculate $y_i^f = (r_1g_1 + r_2g_2 + \cdots + r_mg_m)^f = r_1g_1^f + r_2g_2^f + \cdots + r_mg_m^f$.
Because F_p is abelian, $y_i^f = r_1g_1 + r_2g_2 + \cdots + r_mg_m = y_i$, so $s^{g_k} = y_i^{x^m} \in S_y$.

(5) By part (4), $(S_y \cdot \mathbb{Z}_pG_p)^{n_p} = S_y^{n_p} \cdot (\mathbb{Z}_pG_p)^{n_p} = (S_y)^{n_p} \cdot \mathbb{Z}_pG_p = 0 \cdot \mathbb{Z}_pG_p = 0$.

(6) $(S_y \cdot \mathbb{Z}_pG_p)\mathbb{Z}_pG_p \subset S_y \cdot \mathbb{Z}_pG_p$ and by part (4),
$\mathbb{Z}_pG_p(S_y \cdot \mathbb{Z}_pG_p) = \mathbb{Z}_pG_p(\mathbb{Z}_pG_p \cdot S_y) \subset \mathbb{Z}_pG_p \cdot S_y = S_y \cdot \mathbb{Z}_pG_p$

(7) Since each $y_i^{x^j} \cdot x^i$ is in $S_y \cdot \mathbb{Z}_pG_p$, so also is
$y = \sum_{0\le i<p} x^i \cdot y_i = \sum_{0\le i<p} y_i^{x^{-i}} \cdot x^i$.

(8) We know

$$\begin{aligned}
(\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^{n_p} &\subset (\mathbb{Z}_pG_p(S_y \cdot \mathbb{Z}_pG_p)\mathbb{Z}_pG_p)^{n_p} \\
&\subset (\mathbb{Z}_pG_pS_y \cdot \mathbb{Z}_pG_p\mathbb{Z}_pG_p)^{n_p} \\
&= (\mathbb{Z}_pG_pS_y \cdot \mathbb{Z}_pG_p)^{n_p} \\
&= (S_y \cdot \mathbb{Z}_pG_p\mathbb{Z}_pG_p)^{n_p} \\
&= (S_y \cdot \mathbb{Z}_pG_p)^{n_p}
\end{aligned} \tag{101}$$

By part (5), we conclude that $(\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^{n_p} = 0$.

(9) This clearly follows from part (8)

(10) We need to show that there is an n-stable series for ad_y. We define the series $R_1, \ldots, R_{n_p}$ by $R_k = (\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^k$ for $1 \le k \le n_p$. We notice for $i > j$ that

$(\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^i(\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^j \subseteq (\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^{i+j} \subseteq (\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^i$. Hence for $i > j$, the Lie commutator $((\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^i, (\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^j)$ is a sum and difference of elements of $(\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^i$, so $((\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^i, (\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^j) \subset (\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^i$. Similarly, $((\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^i, y) \subseteq (\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^{i+1}$ because $y \in \mathbb{Z}_pG_py\mathbb{Z}_pG_p$. However, it follows from (8) that $(\mathbb{Z}_pG_py\mathbb{Z}_pG_p)^{n_p} = 1$, so R_p is n_p-Baer.

(11) Corollary 3.3.5 of [**68**] states that $\mathbb{Z}_pG_p/R_p = \mathbb{Z}_pG_p/\Delta(G_p, F_p) \cong \mathbb{Z}_p(G_p/F_p) \cong \mathbb{Z}_pC_p$, where C_p is the group of integers mod p, in this case $\langle x\rangle$.

(12) It follows from (10) and the fact that $\mathbb{Z}_p\langle x\rangle$ is a commutative ring that $\mathbb{Z}_pG_p/R_p$ is commutative.

(13) We showed in part (6) of Lemma 5.16 that $\mathbb{Z}_pG_p$ is not Lie-solvable, but in part (11) we showed that $\mathbb{Z}_pG_p/R_p$ is Lie-solvable. By Lemma 5.4, R_p is not Lie-solvable.

$\square$

CONSTRUCTION 4 (Construction of H_p). *(Rips and Shalev [**6**]) For each prime $p \neq 2$, let H_p be the multiplicative group $1 + R_p$.*

LEMMA 5.19. *(Rips and Shalev [**6**]) The following hold:*

(1) *H_p is a subset of $U(\mathbb{Z}_pG_p)$, the group of units of $\mathbb{Z}_pG_p$.*
(2) *H_p is a subgroup of $U(\mathbb{Z}_pG_p)$.*
(3) *H_p is a p-group of exponent at most p^3.*
(4) *For each $h \in H_p$, the normal closure h^{H_p} of h in H_p is nilpotent of class less than n_p.*
(5) *$U(\mathbb{Z}_pG_p)$ is not a solvable group.*
(6) *$U(\mathbb{Z}_pG_p)/H_p$ is an abelian group.*
(7) *H_p is not a solvable group.*
(8) *H_p is a locally finite group.*

(9) *If $h_1, \ldots, h_d$ are elements of H_p, then the subgroup of H_p generated by $h_1, \ldots, h_d$ has the following properties:*

(a) *it is the product of the cyclic subgroups generated by the h_i;*

(b) *it is normal in H_p;*

(c) *it is nilpotent of class less than or equal to $d(n_p - 1)$.*

(10) *H_p has property $\mathfrak{D}$.*

PROOF. (1) We first note that $R_p \subset \mathbb{Z}_p G_p$. Let $r \in R_p$. Because $(1-(-r))(1+(-r)+(-r)^2+\cdots+(-r)^{n_p-1}) = 1-(-r)^{n_p} = 1+r^{n_p} = 1$, the element $1+r$ has an inverse in $\mathbb{Z}_p G_p$, and so is a unit of $\mathbb{Z}_p G_p$.

(2) Note that the product $(1+r_1)(1+r_2) = 1+(r_1+r_2+r_1r_2)$ is in H_p. Also, the inverse of $1+r$ shown in the proof of (1) is also in H_p. Thus H_p is a subgroup of $U(\mathbb{Z}_p G_p)$.

(3) Let $r \in R_p$, so $1+r \in H_p$. Then

$$\begin{aligned}(1+r)^{p^3} &= \sum_{i=0}^{p^3} \binom{p^3}{i} r^i \\ &= 1 + r^{p^3} \\ &= 1 + r^{p^3-p^2+1} r^{p^2-1} \\ &= 1 + r^{n_p} r^{p^2-1}.\end{aligned} \tag{102}$$

In part (8) of Lemma 5.18, we showed that for any $r \in R_p$, the power r^{n_p} is trivial. Hence

$$\begin{aligned}(1+r)^{p^3} &= 1 + 0 \cdot r^{p^2-1} \\ &= 1.\end{aligned} \tag{103}$$

Therefore H_p has exponent dividing p^3.

(4) Let h be an element of H_p. Consider the commutator $[k_1, k_2, \ldots, k_{n_p}]$, where each k_i is a conjugate of h, that is, $k_i = h^{h_i}$ for some $h_i \in H_p$. Each h_i is of the form $1 + r_i$, where $r_i \in R_p$, so

$$\begin{aligned}
k_i &= (1+r)^{(1+r_i)} \\
&= (1+r_i)^{-1}(1+r)(1+r_i) \\
&= ((1+r_i)^{-1} + (1+r_i)^{-1}r)(1+r_i) \\
&= (1+r_i)^{-1}(1+r_i) + (1+r_i)^{-1}r(1+r_i) \\
&= 1 + (1+r_i)^{-1}r(1+r_i) \\
&= 1 + r^{(1+r_i)}.
\end{aligned} \tag{104}$$

Therefore

$$[k_1, k_2, \ldots, k_{n_p}] = [1 + r^{(1+r_1)}, 1 + r^{(1+r_2)}, \ldots, 1 + r^{(1+r_{n_p})}]. \tag{105}$$

By Lemma 3.1,

$$[k_1, k_2, \ldots, k_{n_p}] = 1 + (r^{(1+r_1)}, r^{(1+r_2)}, \ldots, r^{(1+r_{n_p})}) + b, \tag{106}$$

where b is a sum of terms of higher weight and $(\ ,\ )$ is the algebra commutator in R_p. Now we examine the ring commutator $(r^{(1+r_1)}, r^{(1+r_2)}, \ldots, r^{(1+r_{n_p})})$. Because we defined (r, s) to be $rs - sr$, the commutator $[k_1, k_2, \ldots, k_{n_p}]$ is a sum of terms of the form $r^{(1+r_{\sigma(1)})}r^{(1+r_{\sigma(2)})} \cdots r^{(1+r_{\sigma(n_p)})}$, where $\sigma \in Sym_{n_p}$, the symmetric group on n_p elements. Each $r^{(1+r_{\sigma(i)})}$ is in $\mathbb{Z}_pG_pr\mathbb{Z}_pG_p$ because $R_p \subset \mathbb{Z}_pG_p$. Therefore each $r^{(1+r_{\sigma(1)})}r^{(1+r_{\sigma(2)})} \cdots r^{(1+r_{\sigma(n_p)})}$ is in $(\mathbb{Z}_pG_pr\mathbb{Z}_pG_p)^{n_p}$. However, $(\mathbb{Z}_pG_pr\mathbb{Z}_pG_p)^{n_p} = 0$, so $r^{(1+r_{\sigma(1)})}r^{(1+r_{\sigma(2)})} \cdots r^{(1+r_{\sigma(n_p)})} = 0$. The same argument also shows that the higher power terms in the sum b are trivial. Therefore we conclude that $[k_1, k_2, \ldots, k_{n_p}] = 1$ and hence h^{H_p} is nilpotent of class less than n_p.

(5) We showed in the proof of part (4) of Lemma 5.16 that the commutator subgroup of G_p is not finite, so for $p > 3$, Lemma 5.13 shows that $U(\mathbb{Z}_p G_p)$ is not solvable. If $p = 3$, we need only notice that the Sylow 3-subgroup of G_p is all of G_p, which is not finite.

(6) In part (11) of Lemma 5.18, we showed that $\mathbb{Z}_p G_p / R_p$ is commutative. Hence $U(\mathbb{Z}_p G_p)/(1 + R_p)$ is an abelian group.

(7) If H_p were solvable, then $U(\mathbb{Z}_p G_p)$, the extension of the solvable group H_p by an abelian group, would also be solvable, which contradicts part (5).

(8) Let B be a finitely generated subgroup of H_p. By Parts (3) and (4), B is a finitely generated nilpotent p-group of finite exponent. Thus, by Proposition 5.7, B is finite, so H_p is locally finite.

(9) By Fitting's Theorem the product of d normal nilpotent subgroups of class less than or equal to $n_p - 1$ is a normal nilpotent subgroup of class less than or equal to $d(n_p - 1)$.

(10) This follows immediately from Proposition 5.8 and parts (3) and (9).

□

CONSTRUCTION 5 (Construction of M_p). *(Rips and Shalev* [**6**]*) For each prime $p \neq 2$, let M_p be the wreath product $H_p \wr C_p$. The base group B of $M_p = H_p \wr C_p$ is $\Pi_{i=1}^{p} H_{p,i}$, where each $H_{p,i}$ is an isomorphic copy of H_p via the natural isomorphism $\phi_i : H_p \to H_{p,i}$. For h in H_p, let $h_i = \phi_i(h)$. The action of x, a generator of C_p, on B is given by $h_i^x = h_{i+1}$ for $1 \leq i \leq p-1$ and $h_p^x = h_1$.*

LEMMA 5.20. *(Rips and Shalev* [**6**]*) The following hold:*

(1) *M_p has property $\mathfrak{D}$.*

(2) *M_p is locally nilpotent.*

(3) *M_p is n-Engel for some positive integer n.*

(4) *M_p is not a Baer group.*

PROOF. (1) Lemma 5.19 part (10) and Corollary 5.10 show that M_p has property $\mathfrak{D}$.

(2) By part (1), any finitely generated subgroup of M_p is finite, and since every finite p-group is nilpotent, it follows that M_p is locally nilpotent.

(3) By part (1), M_p has property $\mathfrak{D}$. Hence every 2-generator subgroup of M_p has bounded order. Thus, every 2-generator subgroup has bounded nilpotence class n, for some positive integer n. If b and c are elements of M_p, the commutator $[b,_n c]$ is a commutator of weight $n+1$ in $\langle b, c\rangle$. Thus $[b,_n c]$ is trivial, so M_p is n-Engel.

(4) Assume that M_p is a Baer group, so every cyclic subgroup of M_p is subnormal in M_p. In particular, $\langle x\rangle$ is subnormal in M_p. Thus there is a decreasing sequence of subgroups $M_p = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_m = \langle x\rangle$. In the following we will at times identify x with $((1,\ldots,1),x)$. By 13.1.1 from [**64**] we may assume that $N_i = x^{N_{i-1}}$. Now $[N_{i+1}, N_i] = N_{i+1}^{-1}N_{i+1}^{N_i}$, so $[N_{i+1}, N_i] \subseteq N_{i+1}$. Similarly, $[x, N_i] = x^{-1}x^{N_i} = x^{-1}N_{i+1} \subseteq N_{i+1}$.

We now show that $H_p^{(m)}$ is trivial. We start by showing that $((H_p', 1, \ldots, 1), 1)$ is in N_1. Let h be in H_p. Now

$$\begin{aligned}
[x, ((h^{-1}, 1, \ldots, 1), 1)] &= ((h^{-1}, 1, \ldots, 1), 1)^{-x}((h^{-1}, 1, \ldots, 1), 1)\\
&= ((h, 1, \ldots, 1), 1)^{x}((h^{-1}, 1, \ldots, 1), 1)\\
&= ((1, h, 1, \ldots, 1), 1)((h^{-1}, 1, \ldots, 1), 1)\\
&= ((1, h, 1, \ldots, 1), 1)((h^{-1}, 1, \ldots, 1), 1)\\
&= ((h^{-1}, h, 1, \ldots, 1), 1),
\end{aligned} \tag{107}$$

and so $((h^{-1}, h, 1, \ldots, 1), 1) \in [M_p, x] \subseteq N_1$. Because N_1 is normal in N_0, the conjugate of an element of N_1 by an element of N_0 is in N_1. Thus, for k in H_p,

$$((h^{-1}, h, 1, \ldots, 1), 1)^{((1,k,1,\ldots,1),1)}$$

$$\begin{aligned}
&= ((1,k,1,\ldots,1),1)^{-1}((h^{-1},h,1,\ldots,1),1)((1,k,1,\ldots,1),1) \\
&= ((h^{-1},k^{-1}hk,1,\ldots,1),1) \qquad (108) \\
&= ((h^{-1},h^k,1,\ldots,1),1) \\
&\in N_1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
[x,((h^k,1,\ldots,1),1)] &= ((h^k,1,\ldots,1),1)^{-x}((h^k,1,\ldots,1),1) \\
&= ((h^{-k},1,\ldots,1),1)^{x}((h^k,1,\ldots,1),1) \\
&= ((1,h^{-k},1,\ldots,1),1)((h^k,1,\ldots,1),1) \qquad (109) \\
&= ((h^k,h^{-k},1,\ldots,1),1) \\
&\in N_1,
\end{aligned}$$

so

$$\begin{aligned}
((h^{-1},h^k,1,\ldots,1),1)((h^k,h^{-k},1,\ldots,1),1) &= ((h^{-1}h^k,h^kh^{-k},1,\ldots,1),1) \\
&= (([h,k],1,\ldots,1),1) \qquad (110) \\
&\in N_1.
\end{aligned}$$

Hence $((H_p',1,\ldots,1),1)$ is a subset of N_1.

Now let b and c be elements of $H_p^{(i-1)}$ and assume that $((H_p^{(i-1)},1,\ldots,1),1) \subseteq N_{i-1}$. Then $((b,1,\ldots,1),1)$ is an element of N_{i-1}. Hence,

$$\begin{aligned}
[x,((b^{-1},1,\ldots,1),1)] &= ((b^{-1},1,\ldots,1),1)^{-x}((b^{-1},1,\ldots,1),1) \\
&= ((1,b,1,\ldots,1),1)((b^{-1},1,\ldots,1),1) \qquad (111) \\
&= ((b^{-1},b,1,\ldots,1),1),
\end{aligned}$$

so $((b^{-1},b,1,\ldots,1),1) \in [N_{i-1},x] \subseteq N_i$. Moreover,

$$
\begin{aligned}
&((b^{-1}, b, 1, \ldots, 1), 1)^{((1,c,1,\ldots,1),1)} \\
&= ((1, c, 1, \ldots, 1), 1)^{-1}((b^{-1}, b, 1, \ldots, 1), 1)((1, c, 1, \ldots, 1), 1) \\
&= ((1, c^{-1}, 1, \ldots, 1), 1)^{-1}((b^{-1}, b, 1, \ldots, 1), 1)((1, c, 1, \ldots, 1), 1) \\
&= ((b^{-1}, b^{c}, 1, \ldots, 1), 1) \\
&\in N_i.
\end{aligned} \tag{112}
$$

By the induction hypothesis, $((b^c, 1, \ldots, 1), 1)$ is in N_{i-1}. This means

$$
\begin{aligned}
[x, ((b^c, 1, \ldots, 1), 1)] &= ((b^c, 1, \ldots, 1), 1)^{-x}((b^c, 1, \ldots, 1), 1) \\
&= ((1, b^{-c}, 1, \ldots, 1), 1)((b^c, 1, \ldots, 1), 1) \\
&= ((b^c, b^{-c}, 1, \ldots, 1), 1) \\
&\in N_i,
\end{aligned} \tag{113}
$$

so

$$
\begin{aligned}
((b^{-1}, b^c, 1, \ldots, 1), 1)((b^c, b^{-c}, 1, \ldots, 1), 1) &= ((b^{-1}b^c, b^c b^{-c}, 1, \ldots, 1), 1) \\
&= (([b, c], 1, \ldots, 1), 1) \\
&\in N_i.
\end{aligned} \tag{114}
$$

Therefore $((H_p^{(i)}, 1, \ldots, 1), 1)$ is a subset of N_i. By induction, $((H_p^{(j)}, 1, \ldots, 1), 1) \subseteq N_j$ for all j, $1 \le j \le m$, and so $((H_p^{(m)}, 1, \ldots, 1), 1) \subseteq ((1, \ldots, 1)\langle x \rangle)$. Thus $H_p^{(m)}$ is trivial. This contradicts the fact that H_p is not solvable, and hence M_p is not a Baer group.

□

Proof of Theorem 5.15. Let p be an odd prime. We showed in part (3) of Lemma 5.20 that M_p is n-Engel for some positive integer n. Assume for the sake of contradiction that for every positive integer $i \le n$, the normal closure of every element of an i-Engel p-group is $(i-1)$-Engel. We show that this assumption leads to a contradiction of known properties of M_p. Let m be an element of M_p. Let N_1 be the normal closure of m in M_p. We then inductively define N_s to be $m^{N_{s-1}}$ for $s = 2, \dots, n$. The subgroup N_1 is normal in M_p and, by our assumption, is $(n-1)$-Engel. If N_{s-1} is $(n-s+1)$-Engel, the induction hypothesis says that N_s is $(n-s)$-Engel. Thus, each N_s is $(n-s)$-Engel and normal in N_{s-1}, so subnormal in M_p. Then N_{n-1} is subnormal in M_p, and is $(n-(n-1))$-Engel, meaning that it is abelian. Hence N_n is an abelian subgroup that contains m and is subnormal in M_p. Since $\langle m \rangle$ is normal in N_n, we conclude that $\langle m \rangle$ is subnormal of defect $n+1$ in M_p. Since this argument holds for every element m in M_p, we have proved that M_p is Baer, which contradicts part (4) of Lemma 5.20. $\square$

2.4. Construction for $p = 2$. Rips and Shalev make the comment that although their construction works only for odd primes, it can be modified to work for $p = 2$ also. In this subsection we show how to make the necessary modifications to the proof.

THEOREM 5.21. *For some positive integer n, there is an n-Engel 2-group that has an element whose normal closure in the whole group is not $(n-1)$-Engel.*

The construction of such a group is very similar to that of Rips and Shalev's construction for odd primes, and in fact follows from a hint in their paper [**6**].

CONSTRUCTION 6 (Construction of G_2). *Let* $I_2 = \langle a_i, z_i \mid a_i^4 = z_i^4 = [a_i, a_j] = [a_i, z_j] = [z_i, z_j] = 1 \text{ for } i, j \in \{1, 2, \dots\} \rangle$, *so I_2 is $(C_4 \times C_4)^\omega$. Let X_2 be the group $\langle x \mid x^4 = 1 \rangle$ and let G_2 be the group $I_2 \rtimes X_2$, where the action of X_2 on I_2 is defined by $a_i^x = a_i z_i$ and $z_i^x = z_i$.*

LEMMA 5.22. *The following hold:*

(1) *G_2 is nilpotent of class 2.*

(2) *G_2 is of exponent 8.*

(3) *G_2 has an abelian subgroup of index 4.*

(4) *G_2 does not have a 2-abelian subgroup of index at most 2.*

(5) *$\mathbb{Z}_2 G_2$ is not a Lie-solvable ring.*

Proof. (1) The proof of this is exactly the same as in part (1) of Lemma 5.16.

(2) We compute the eighth power of an element of G_2. By Lemma 5.3 and the action of X_2 on I_2, if w_k, v_k, and j are all integers,

$$\begin{aligned}((a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \dots), x^j)^8 &= ((a_1^{w_1} z_1^{v_1} a_2^{w_1} z_2^{v_2} \dots)(a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \dots)^{x^j} \\ &\quad \dots (a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \dots)^{(x^j)^7}, (x^j)^8) \\ &= ((a_1^{w_1} z_1^{v_1} a_2^{w_2} z_2^{v_2} \dots)(a_1^{w_1} z_1^{(1+j)v_1} a_2^{w_2} z_2^{(1+j)v_2} \dots) \\ &\quad \dots (a_1^{w_1} z_1^{(1+7j)v_1} a_2^{w_2} z_2^{(1+7j)v_2} \dots), x^{8j}) \\ &= ((a_1^{8w_1} z_1^{(8+8j(7)/2)v_1} a_2^{8w_2} z_2^{(8+8j(7)/2)v_2} \dots), 1) \\ &= ((a_1^{8w_1} z_1^{(8+4j(7))v_1} a_2^{8w_2} z_2^{(8+4j(7))v_2} \dots), 1).\end{aligned} \tag{115}$$

Because the a_i and the z_i are of order 4, we conclude that G_2 is of exponent at most 8. Because $(a_1 z_1, x)^4 = (a_1^4 z_1^{10}, x^4) = (z_1^2, 1)$ is not trivial, the exponent of G_2 is greater than 4, and so is exactly 8.

(3) The subgroup $((C_4 \times C_4)^{\omega}, 1)$ is of index 4 in G_2 and is abelian.

(4) In the proof of part (1) of Lemma 5.16, we computed the commutator of any two elements of G_p for odd prime p. The same calculation holds in G_2. Thus $\langle z_i \rangle$ contains G_2'. Each $(z_i, 1)$ is in G_2' because $[a_i, x] = z_i$. Hence G_2' is the subgroup $\langle z_i \rangle$ of G_2, and so is infinite. Therefore G_2 is not 2-abelian.

Assume that W is a 2-abelian subgroup of index 2 in G_2. Because G_2 is a 2-group, W is a subgroup of index 2 in G_2 with finite commutator subgroup W'. We see that $(1, x^2) \in W$ because $(1, x^2) = (1, x)^2$. Suppose for a positive integer i that $(a_i, x) \in W$. Then $[(a_i, 1), (1, x^2)] = (z_i^2, 1)]$ is also in W. The $\{z_k\}$ are generators of a relatively free abelian group of exponent 4, so $(z_i^2, 1) = (z_j^2, 1)$ only when $i = j$. Thus, because W' is finite, we deduce that there are only finitely many i with $(a_i, 1)$ in W. Next, we consider a left coset of W. If $(a_i, 1)$ and $(a_j, 1)$ are both in the same left coset of W, then $(a_i, 1)^{-1}(a_j, 1) = (a_i^{-1}a_j, 1) \in W$. However, this conclusion implies that $[(a_i^{-1}a_j, 1), (1, x^2)] = (z_i^2 z_j^2, 1)]$ is also in W. Thus only finitely many $(a_i, 1)$ are in any one left coset of W. Because there are infinitely many a_i and only finitely many are in W, we conclude that the index of W in G is infinite, which is a contradiction, so no such W exists.

(5) This follows immediately from part (c) of Lemma 5.12.

$\square$

CONSTRUCTION 7 (Construction of R_2). *We define R_2 to be the ring*

$$\mathbb{Z}_2 G_2 \Delta F_2, \tag{116}$$

where $F_2 = FC(G_2)$ and ΔF_2 is the augmentation ideal of $\mathbb{Z}_2 F_2$.

As in the odd prime case, $\Delta(G_2, F_2)$ is the left ideal of $\mathbb{Z}_2 G_2$ generated by $\{h - 1 \mid h \in F_2\}$.

LEMMA 5.23. *The following hold:*

(1) *F_2 is abelian.*

(2) *$G_2 \cong F_2 \rtimes C_4$.*

(3) *$a^8 = 0$ for every $a \in \Delta(F_2)$.*

(4) *$R_2 = \Delta(G_2, F_2)$.*

(5) *$R_2 = \bigoplus_{0 \le i < 4} x^i \cdot \Delta F_2$.*

PROOF. (1) The proof that $F_2 = ((C_4 \times C_4)^\omega, 1) = (I_2, 1)$ is exactly the same as for part (1) of Lemma 5.17.

(2) The proof is identical to the proof of part (2) of Lemma 5.17

(3) By the definition of augmentation ideal, any element a of ΔF_2 is of the form $r_1 g_1 + r_2 g_2 + \cdots + r_m g_m$, where $r_i \in \mathbb{Z}_2$, $g_i \in G_2$, and $\sum r_i = 1$. In a ring over $\mathbb{Z}_2$, $(i + j)^p = i^8 + j^8$, so

$$
\begin{aligned}
a^8 &= (r_1g_1 + r_2g_2 + \cdots + r_{m-1}g_{m-1} + r_mg_m)^8 \\
&= (r_1g_1)^8 + (r_2g_2)^8 + \cdots + (r_{m-1}g_{m-1})^8 + (r_mg_m)^8 \\
&= r_1^8 1_G + r_2^8 1_G + \cdots + r_{m-1}^8 1_G + r_m^8 1_G \\
&= r_1 1_G + r_2 1_G + \cdots + r_{m-1} 1_G + r_m 1_G \\
&= (r_1 + r_2 + \cdots + r_{m-1} + r_m)1_G.
\end{aligned} \tag{117}
$$

Since $\sum r_i = 0$, we conclude that $a^8 = 0$.

(4) By part (i) of exercise 2 on page 138 of [**68**], $R_2 = \Delta(G_2, F_2)$.

(5) We first note that $\{x^i \mid i \in \{0, \ldots, 3\}\}$ is a transversal of F_2 in G_2. By Proposition 3.3.3 of [**68**], $\{x^i(h-1) \mid i \in \{0, \ldots, p-1\}, h \in F_2, h \neq 1\}$ is a basis for $\Delta(G_2, F_2)$. However, $\Delta(G_2, F_2)$ is defined as the left ideal of $\mathbb{Z}_2G_2$ generated by $\{h - 1 \mid h \in F_2\}$. This gives the desired conclusion.

□

By part (5) of Lemma 5.23, every $y \in R_2$ may be written uniquely in the form $\sum_{0 \le i < 4} x^i \cdot y_i$, where $y_i \in \Delta F_2$. Let $S_y = \{y_i^{x^j} : 0 \le i, j < 4\}$. Also let $n_2 = 7 \cdot 16 + 1 = 113$.

Lemma 5.24. *Let $y \in R_2$. Then the following hold:*

(1) $b^8 = 0$ *for every element* $b \in S_y$.

(2) $| S_y | \le 16$.

(3) $S_y^{n_2} = 0$.

(4) $S_y \cdot \mathbb{Z}_2G_2 = \mathbb{Z}_2G_2 \cdot S_2$.

(5) $(S_y \cdot \mathbb{Z}_2G_2)^{n_2} = (S_2)^{n_2} \cdot \mathbb{Z}_2G_2 = 0$.

(6) $S_y \cdot \mathbb{Z}_2G_2$ *is a 2-sided ideal of* $\mathbb{Z}_2G_2$.

(7) $y \in S_y \cdot \mathbb{Z}_2G_2$.

(8) $(\mathbb{Z}_2G_2 y \mathbb{Z}_2G_2)^{n_2} = 0$.

(9) $r^{n_2} = 0$ *for every* r *in* R_2.

(10) R_2 *is Baer-bounded.*

(11) $\mathbb{Z}_2G_2/R_2 \cong \mathbb{Z}_2\langle x\rangle$ *as rings.*

(12) $\mathbb{Z}_2G_2/R_2$ *is a commutative ring.*

(13) R_2 *is not a Lie-solvable ring.*

PROOF. (1) Any $b \in S_y$ is of the form $y_i^{x^j}$, where y_i is in $\Delta(F_2)$, so $b^8 = (y_i^{x^j})^8 = (y_i^8)^{x^j} = 0$.

(2) The elements of S_y are made by making two choices, each of which has four possibilities.

(3) We want to show that any product of n_2 elements of S_y is trivial. We first note that F_2 is abelian, and that any conjugate of an element of F_2 by x remains in F_2. Thus the elements of S_y commute. Because $b^8 = 0$ for each $b \in S_y$, a product that contains any factor eight or more times is trivial. Since S_y has at most 16 elements, this gives 16(7) as the largest number of factors in a nontrivial product.

(7) Since each $y_i^{x^i} \cdot x^i$ is in $S_y \cdot \mathbb{Z}_2G_2$, so also is $y = \sum_{0\leq i<4} x^i \cdot y_i = \sum_{0\leq i<4} y_i^{x^{-i}} \cdot x^i$.

The proofs of parts (4)-(6) and (8)-(13) are the same as the respective proofs in Lemma 5.18. □

CONSTRUCTION 8 (Construction of H_2). *Let H_2 be the multiplicative group $1 + R_2$.*

LEMMA 5.25. *The following hold:*

(1) *H_2 is a subset of $U(\mathbb{Z}_2G_2)$, the group of units of $\mathbb{Z}_2G_2$.*

(2) *H_2 is a subgroup of $U(\mathbb{Z}_2G_2)$.*

(3) *H_2 is a 2-group of exponent at most 128.*

(4) *For each $h \in H_2$, the normal closure h^{H_2} of h in H_2 is nilpotent of class less than n_2.*

(5) *$U(\mathbb{Z}_2G_2)$ is not a solvable group.*

(6) *$U(\mathbb{Z}_2G_2)/H_2$ is an abelian group.*

(7) *H_2 is not a solvable group.*

(8) *H_2 is a locally finite group.*

(9) *If $h_1, \ldots, h_d$ are elements of H_2, then the subgroup of H_2 generated by $h_1, \ldots, h_d$ has the following properties:*

   (a) *it is the product of the cyclic subgroups generated by the h_i;*

   (b) *it is normal in H_2;*

   (c) *it is nilpotent of class less than or equal to $d(n_2 - 1)$.*

(10) *H_2 has property $\mathfrak{D}$.*

Proof. (1) We first notice that $R_2 \subset \mathbb{Z}_2 G_2$. Let $r \in R_2$. Because $(1 - (-r))(1 + (-r) + (-r)^2 + \cdots + (-r)^{n_2 - 1}) = 1 - (-r)^{n_2} = 1 + r^{n_2} = 1$, the element $1 + r$ has an inverse in $\mathbb{Z}_2 G_2$, and so is a unit of $\mathbb{Z}_2 G_2$.

(2) The proof of this result is the same as the proof for part (2) of Lemma 5.19.

(3) Let $r \in R_2$, so $1 + r$ is in H_2. Then

$$\begin{aligned}
(1+r)^{128} &= \sum_{i=0}^{128} \binom{128}{i} r^i \\
&= 1 + r^{128} \\
&= 1 + r^{128 - 7\cdot 16 + 1} r^{7\cdot 16 - 1} \\
&= 1 + r^{n_2} r^{7\cdot 16 - 1} \\
&= 1 + 0 \cdot r^{7\cdot 16 - 1} \\
&= 1.
\end{aligned} \tag{118}$$

Hence H_2 has exponent dividing 128.

(4) The proof of this result is the same as the proof for part (4) of Lemma 5.19.

(5) Assume that $U(\mathbb{Z}_2 G_2)$ is solvable. Then because G_2 is a 2-group, by Lemma 5.14, G_2 contains a finite normal subgroup N such that either G/N is abelian or else G/N contains an abelian normal subgroup of index 2 and $(G/N)/Z(G/N)$ has bounded exponent. We first ask what elements of G_2 can be in N. Because X_2 and I_2 are abelian, for $m, n \in I_2$ and $s, t \in X_2$, by

Lemma 5.2,

$$(m,s)^{(n,t)} = ((n^{-1}mn^{s})^{t^{-1}}, s). \tag{119}$$

Now if $m = (a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)$ and $n = (a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)$, and if also $s = x^i$ and $t = x^j$, then

$$\begin{aligned}
((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots), x^i)^{((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots), x^j)} &= (((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-1} \\
&\qquad \cdot (a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots) \\
&\qquad \cdot (a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{x^i})^{x^{-j}}, x^i) \\
&= (((a_1^{-\hat{v}_1} z_1^{-\hat{w}_1} a_2^{-\hat{v}_2} z_2^{-\hat{w}_2} \dots) \\
&\qquad \cdot (a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots) \\
&\qquad \cdot (a_1^{\hat{v}_1} z_1^{\hat{w}_1 + i\hat{v}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2 + i\hat{v}_2} \dots))^{x^{-j}}, x^i) \\
&= (((a_1^{v_1 - \hat{v}_1} z_1^{w_1 - \hat{w}_1} a_2^{v_2 - \hat{v}_2} z_2^{w_2 - \hat{w}_2} \dots) \\
&\qquad \cdot (a_1^{\hat{v}_1} z_1^{\hat{w}_1 + i\hat{v}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2 + i\hat{v}_2} \dots))^{x^{-j}}, x^i) \\
&= ((a_1^{v_1} z_1^{w_1 + i\hat{v}_1} a_2^{v_2} z_2^{w_2 + i\hat{v}_2} \dots)^{x^{-j}}, x^i) \\
&= ((a_1^{v_1} z_1^{w_1 + i\hat{v}_1 - jv_1} a_2^{v_2} z_2^{w_2 + i\hat{v}_2 - jv_2} \dots), x^i).
\end{aligned} \tag{120}$$

If i is not zero, then m has infinitely many conjugates. Because N is normal and finite, we conclude that i is zero for all m in H_2, so N must be a finite subgroup of I_2. We know I_2 is abelian, and G_2 is not abelian, so G_2/N cannot be abelian. Thus G_2/N contains an abelian normal subgroup B of index 2. Let $\phi : G \to G/N$ be the natural quotient homomorphism.

Any two elements of B are elements of G_2/N, so they can be assumed to be $\phi(m,s) = \phi((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots), x^i)$ and $\phi(n) = \phi((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots), x^j)$. Then using Lemma 5.2,

$$\begin{aligned}
[\phi(m), \phi(n)] &= \phi([(m,s),(n,t)]) \\
&= \phi((m^{-1} m^{t^{-1}})^{s^{-1}} (n^{-s^{-1}} n)^{t^{-1}}, 1) \\
&= \phi((((a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)^{-1} (a_1^{v_1} z_1^{w_1} a_2^{v_2} z_2^{w_2} \dots)^{x^{-j}})^{x^{-i}} \\
&\qquad ((a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots)^{-x^{-i}} (a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots))^{x^{-j}}, 1)
\end{aligned}$$

$$
\begin{aligned}
&= \phi(((a_1^{-v_1} z_1^{-w_1} a_2^{-v_2} z_2^{-w_2} \dots)(a_1^{x^{-j}v_1} z_1^{x^{-j}w_1} a_2^{x^{-j}v_2} z_2^{x^{-j}w_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-x^{-i}\hat{v}_1} z_1^{-x^{-i}\hat{w}_1} a_2^{-x^{-i}v_2} z_2^{-x^{-i}\hat{w}_2} \dots)(a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots))^{x^{-j}}, 1) \\
&= \phi(((a_1^{-v_1} z_1^{-w_1} a_2^{-v_2} z_2^{-w_2} \dots)(a_1^{x^{-j}v_1} z_1^{w_1} a_2^{x^{-j}v_2} z_2^{w_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-x^{-i}\hat{v}_1} z_1^{-\hat{w}_1} a_2^{-x^{-i}\hat{v}_2} z_2^{-\hat{w}_2} \dots)(a_1^{\hat{v}_1} z_1^{\hat{w}_1} a_2^{\hat{v}_2} z_2^{\hat{w}_2} \dots))^{x^{-j}}, 1) \qquad (121) \\
&= \phi(((a_1^{-v_1} a_2^{-v_2} \dots)(a_1^{x^{-j}v_1} a_2^{x^{-j}v_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-x^{-i}\hat{v}_1} a_2^{-x^{-i}\hat{v}_2} \dots)(a_1^{\hat{v}_1} a_2^{\hat{v}_2} \dots))^{x^{-j}}, 1) \\
&= \phi(((a_1^{-v_1} a_2^{-v_2} \dots)(a_1^{v_1} z_1^{-jv_1} a_2^{v_1} z_2^{-jv_2} \dots))^{x^{-i}} \\
&\quad ((a_1^{-\hat{v}_1} z_1^{i\hat{v}_1} a_2^{-\hat{v}_2} z_2^{i\hat{v}_2} \dots)(a_1^{\hat{v}_1} a_2^{\hat{v}_2} \dots))^{x^{-j}}, 1) \\
&= \phi((z_1^{-jv_1} z_2^{-jv_2} \dots)^{x^{-i}} (z_1^{i\hat{v}_1} z_2^{i\hat{v}_2} \dots)^{x^{-j}}, 1) \\
&= \phi((z_1^{-jv_1} z_2^{-jv_2} \dots)(z_1^{i\hat{v}_1} z_2^{i\hat{v}_2} \dots), 1) \\
&= \phi((z_1^{i\hat{v}_1 - jv_1} z_2^{i\hat{v}_2 - jv_2} \dots), 1).
\end{aligned}
$$

We know that B is abelian, so either j is zero, or else $\phi(z_i)$ is trivial for every i. However, N is finite, so $\phi(z_i)$ can only be trivial for finitely many i. We conclude that every element of B is of the form $\phi((m, 1))$. However, $\langle x \rangle$ is of order 4 in G_2, so $\langle \phi(x) \rangle$ is of order 4 in G_2/N. This contradicts our assertion that the index of B in G_2/N is 2. We conclude that $U(\mathbb{Z}_2 G_2)$ is not solvable.

The proofs of the remaining parts are the same as the proofs of parts (6) to (10) of Lemma 5.19. □

Construction 9 (Construction of M_2). *Let M_2 be the wreath product $H_2 \wr C_2$. The base group B of $M_2 = H_2 \wr C_2$ is $\Pi_{i=1}^{2} H_{2,i}$, where each $H_{2,i}$ is an isomorphic copy of H_2 with the natural isomorphism $\phi_i : H_2 \to H_{2,i}$. For h in H_2, let $h_i = \phi_i(h)$. The action of x, a generator of C_2, on B is given by $h_1^x = h_2$ and $h_2^x = h_1$.*

Lemma 5.26. *The following claims hold.*

(1) *M_2 has property $\mathfrak{D}$.*

(2) *M_2 is locally nilpotent.*

(3) *M_2 is n-Engel for some positive integer n.*

(4) M_2 *is not a Baer group.*

PROOF. The proof is identical to that for Lemma 5.20. $\square$

PROOF OF THEOREM 5.21. The proof of Theorem 5.15 also applies here. $\square$

3. Bounds

3.1. Bounds for Odd Primes. In this subsection we determine for each prime p an upper bound for the least n for which the normal closure of an element of an n-Engel group need not be $(n-1)$-Engel. We do so by showing how to get such a bound for p-groups for each odd prime p.

PROPOSITION 5.27. *For each prime $p > 2$ there is an n-Engel p-group whose normal closure is not $(n-1)$-Engel for a value of n that is bounded above by:*

$$3p \sum_{k=1}^{2(p-1)p^3} \left(\sum_{d=1}^{\lfloor 2(p-1)p^3/k \rfloor} \left(\frac{\mu(d)}{d} \right) \cdot \frac{(2p)^k}{k} \right). \tag{122}$$

PROOF. If n is an upper bound for the nilpotence class of a 2-generator subgroup of M_p, then we know that M_p is n-Engel since the Engel law is defined on two group elements. Thus we apply Proposition 5.11 with $s = 2$. The exponent of H_p is p^3 and by Lemma 5.19 part (9c), a $2p$-generator subgroup of H_p is nilpotent of class $2(p-1)p^3$, so $\beta_2 = 2(p-1)p^3$ and $e = p^3$. $\square$

The results of calculating the bound of Proposition 5.27 (using Maple) for specific primes are the following: For the prime $p = 3$, the bound is about 1×10^{83}, for $p = 5$ the bound is about 2×10^{998}, and for $p = 7$ the calculated bound is approximately 2×10^{4715}. It would be of interest to know if these bounds can be improved. An indication of how much these bounds may overestimate the actual value comes from comparing the bound of about 1×10^{83} for a 3-group with the actual value of 5 for a 3-group from Section 1 of this chapter.

3.2. Bounds for 2. We find a bound for 2-groups using the same methodology as in the previous subsection.

PROPOSITION 5.28. *There is an n-Engel 2-group whose normal closure is not $(n-1)$-Engel for a value of n that is bounded above by:*

$$14\sum_{k=1}^{448}\left(\sum_{d=1}^{\lfloor 448/k\rfloor}\left(\frac{\mu(d)}{d}\right)\cdot\frac{(4)^k}{k}\right). \tag{123}$$

PROOF. If n is an upper bound on the nilpotence class of a 2-generator subgroup of M_2, then we know that M_2 is n-Engel. Thus, we apply Proposition 5.11 with $s=2$. The exponent of H_2 is 128 and by Lemma 5.25 part (9c), a d-generator subgroup of H_2 is nilpotent of class $d(n_2-1)=d(7\cdot 16)$, so $\beta_d=d(7\cdot 16)$ and $e=128$. Thus n is bounded above by

$$2\left(\sum_{i=1}^{f(4)}\left(\sum_{j=1}^{\lfloor f(4)/i\rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(4)^i}{i}\right)\right)\cdot log_2 128, \tag{124}$$

which simplifies to

$$2\cdot 7\left(\sum_{i=1}^{4\cdot 7\cdot 16}\left(\sum_{j=1}^{\lfloor 4\cdot 7\cdot 16/i\rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(4)^i}{i}\right)\right), \tag{125}$$

or

$$14\left(\sum_{i=1}^{448}\left(\sum_{j=1}^{\lfloor 448/i\rfloor}\left(\frac{\mu(j)}{j}\right)\cdot\frac{(4)^i}{i}\right)\right). \tag{126}$$

$\square$

The value of this bound, according to Maple, is about 2×10^{268}.

Chapter 6

A Class of Groups In Which $\mathfrak{E}_n$ and $\mathfrak{L}(\mathfrak{E}_{n-1})$ are Equivalent

1. Preliminaries

In this chapter we consider a class of solvable groups in which condition $\mathfrak{E}_n$, the n-Engel condition, implies condition $\mathfrak{L}(\mathfrak{E}_{n-1})$, that is, that the normal closure of every element is $(n-1)$-Engel. The results of this chapter generalize Theorem 1.4. We first need a few definitions.

Definition 6.1. If a is an element of a group G and n is a positive integer, then we say that a is a *left n-Engel element* of G if $[g, {}_n a] = 1$ for every $g \in G$.

Definition 6.2. Let $W = \{\theta_a \mid a \in A\}$ be a set of words in variables $x_1, x_2, \ldots$. Let $\mathfrak{A}$ be the class of all groups G such that each θ_a reduces to the identity element when the variables x_i are replaced by arbitrary elements of G. Then $\mathfrak{A}$ is called the *variety determined by the set of words W*. Each θ_a in W is referred to as a *law* of the variety $\mathfrak{A}$.

Two varieties of groups are said to be *equivalent* if they contain the same groups.

We start by defining a variety of groups. For an integer $n \geq 4$, we consider a group G to be a $\mathfrak{W}_n$ group if for every i in $\{3, \ldots, n-1\}$ the law $[x_1, \ldots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \ldots, x_n] = 1$ holds for all $x_1, \ldots, x_n$ in G. In [**7**], Morse proved the results of this section for $\mathfrak{W}_6$ groups and $\mathfrak{W}_7$ groups. We show that there exists a positive integer n such that the variety $\mathfrak{W}_{n+1}$ is not equivalent to the variety $\mathfrak{N}_n$ of groups that are nilpotent of class n.

Lemma 6.1. *The variety $\mathfrak{W}_8$ is not equivalent to the variety $\mathfrak{N}_7$.*

PROOF. If we use the GAP script:

```
RequirePackage("anupq");
f:=FreeGroup(2);
ids:=[function(a,b,c,d,e,f,i,j) return LeftNormedComm(
  [a,b,LeftNormedComm([c,d]),e,f,i,j]); end,
function(a,b,c,d,e,f,i,j) return LeftNormedComm(
  [a,b,c,LeftNormedComm([d,e]),f,i,j]); end,
function(a,b,c,d,e,f,i,j) return LeftNormedComm(
  [a,b,c,d,LeftNormedComm([e,f]),i,j]); end,
function(a,b,c,d,e,f,i,j) return LeftNormedComm(
  [a,b,c,d,e,LeftNormedComm([f,i]),j]); end,
function(a,b,c,d,e,f,i,j) return LeftNormedComm(
  [a,b,c,d,e,f,LeftNormedComm([i,j])]); end];

g:=Pq(f: Prime:=11, Exponent:=11, ClassBound:=8,
Identities:=ids);
```

then GAP returns:

```
#I  Class 1 with 2 generators.
#I  Class 2 with 3 generators.
#I  Class 3 with 5 generators.
#I  Class 4 with 8 generators.
#I  Class 5 with 14 generators.
#I  Class 6 with 23 generators.
#I  Class 7 with 41 generators.
#I  Class 8 with 48 generators.
```

Here, GAP tells us that the largest 11-group on 2 generators in the variety $\mathfrak{W}_8$ is not nilpotent of class 7, because it has class 8 pc-generators. From this we see that $\mathfrak{W}_8$ is not equivalent to $\mathfrak{N}_7$. $\square$

LEMMA 6.2. *The variety of groups defined by the set of laws* $\{[x_1, \dots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \dots, x_n] = 1 \mid i \in \{1, \dots, n-1\}\}$ *is equivalent to* $\mathfrak{N}_{n-1}$.

PROOF. If a group G is nilpotent of class $n-1$, it is clear that G satisfies $[x_1, \dots, x_{i-1}, [x_i, x_{i+1}], x_{i+2}, \dots, x_n] = 1$ for every i in $\{1, \dots, n-1\}$. However, the law for $i = 1$ implies nilpotence of class at most $n-1$, which completes the proof. $\square$

We need a preliminary result.

LEMMA 6.3. *Let G be a group and let a be an element of G. If there is some integer j such that $[a^{g_1}, a^{g_2}, \ldots, a^{g_j}, a] = 1$ for any $g_1, \ldots, g_j \in G$, then a^G is nilpotent of class at most j.*

PROOF. It is sufficient to show that $[a^{h_1}, a^{h_2}, \ldots, a^{h_j}, a^{h_{j+1}}] = 1$, where $h_i \in G$ for $i \in \{1, 2, \ldots, j+1\}$. However,

$$[a^{h_1}, a^{h_2}, \ldots, a^{h_j}, a^{h_{j+1}}] = [a^{h_1 h_{j+1}^{-1}}, a^{h_2 h_{j+1}^{-1}}, \ldots, a^{h_j h_{j+1}^{-1}}, a]^{h_{j+1}}, \tag{127}$$

which is trivial by hypothesis. □

2. Properties of Class $\mathfrak{W}_m$

LEMMA 6.4. *Let G be a $\mathfrak{W}_{t+3}$ group, where t is a positive integer. Let j be a positive integer. Suppose $u \in \gamma_{t-j}(G)$ if $j < t$ or $u \in G$ if $j \geq t$. Also let $v \in G'$. If $a_i \in G$ for $1 \leq i \leq j$, then there exists an element w of G' such that $[uv, a_1, \ldots, a_j] = [u, a_1, \ldots, a_j]^w [v, a_1, \ldots, a_j]$.*

PROOF. This proof is a generalization of the proofs of Lemmas 2 and 2′ in [**7**]. We prove this lemma by induction on j. By Lemma 1.1, $[uv, a_1] = [u, a_1]^v [v, a_1]$, so the result is true for $j = 1$. Now we assume that the result is true for $j \geq 1$ and we wish to show that it is true for $j+1$. Let $u \in \gamma_{t-(j+1)}$ if $j+1 < t$ or $u \in G$ if $j+1 \geq t$, and let $v \in G'$. Now by Lemma 1.1,

$$[uv, a_1, \ldots, a_{j+1}] = [[u, a_1][u, a_1, v][v, a_1], a_2, \ldots, a_{j+1}]. \tag{128}$$

Note that $[u, a_1] \in \gamma_{t-j}(G)$ if $j+1 < t$ or $[u, a_1] \in G'$ if $j+1 \geq t$. In particular, if $j = t-1$, then $[u, a_1] \in G' = \gamma_2(G) \subseteq \gamma_1(G) = \gamma_{t-(t-1)}(G)$. Thus $[u, a_1] \in \gamma_{t-j}(G)$ if $j < t$ or $[u, a_1] \in G'$ if $j \geq t$. Then by the induction hypothesis, there exists $w \in G'$ such that

$$[uv, a_1, \ldots, a_{j+1}] = [[u, a_1][u, a_1, v], a_2, \ldots, a_{j+1}]^w [v, a_1, a_2, \ldots, a_{j+1}]. \tag{129}$$

Using the induction hypothesis again, there is an element $\hat{w} \in G'$ such that

$$[uv, a_1, \dots, a_{j+1}] = [u, a_1, a_2, \dots, a_{j+1}]^{\hat{w}w}[u, a_1, v, a_2, \dots, a_{j+1}]^{w}[v, a_1, a_2, \dots, a_{j+1}]. \tag{130}$$

Now we consider $[u, a_1, v, a_2, \dots, a_{j+1}]$. We know $[u, a_1]$ is a product of simple commutators of length at least $t - j$ (or length 2 if $j \geq t$). By repeated application of Lemma 1.1, because G is a $\mathfrak{W}_{t+3}$ group, and because $v \in G'$, $[u, a_1, v, a_2, \dots, a_{j+1}] = 1$. This completes the induction, and so completes the proof. □

LEMMA 6.5. *Let G be a $\mathfrak{W}_{t+3}$ group, where t is a positive integer, and let j be a positive integer. Suppose that $u \in \gamma_{t-j}(G)$ if $j < t$, or suppose that $u \in G$ if $j \geq t$. Also, suppose that $a_i \in G$ for $1 \leq i \leq j$. Then there exists an element $w \in G'$ such that $[u^{-1}, a_1, \dots, a_j] = [u, a_1, \dots, a_j]^{-w}$.*

PROOF. This proof also follows the proofs of Lemmas 2 and 2′ in [**7**]. By Lemma 6.4 there exists an element w in G' such that
$1 = [u^{-1}u, a_1, \dots, a_j] = [u^{-1}, a_1, \dots, a_j]^{w}[u, a_1, \dots, a_j]$, and so
$[u^{-1}, a_1, \dots, a_j] = ([u, a_1, \dots, a_j]^{w^{-1}})^{-1} = [u, a_1, \dots, a_j]^{-w^{-1}}$. □

LEMMA 6.6. *Let G be a $\mathfrak{W}_{t+3}$ group, where t is a positive integer. Suppose n is a positive integer greater than t. If a is a left $(n+1)$-Engel element of G, then a^G is nilpotent of class at most n.*

PROOF. This proof also follows the proofs of Lemmas 3 and 3′ in [**7**]. We first prove by induction on k, for $1 \leq k \leq n$, that $[a^{g_1}, \dots, a^{g_k},_{n+1-k} a] = 1$ for all $g_1, \dots, g_k$ in G. For $k = 1$ we notice that

$$[a^g,_n a] = [a[a, g],_n a]. \tag{131}$$

Because $n > t$, we can apply Lemma 6.4 to see that for some $w \in G'$,

$$
\begin{aligned}
[a^g, {}_n a] &= [a, {}_n a]^w [[a, g], {}_n a] \\
&= [a, g, {}_n a] \\
&= [[g, a]^{-1}, {}_n a].
\end{aligned} \tag{132}
$$

Clearly $[g, a] \in G$. By Lemma 6.5, we conclude that $[a^g, {}_n a]$ is equal to $[g, {}_{n+1} a]^{-w}$ for some $w \in G'$, but $[g, {}_{n+1} a]$ is trivial since a is a left $(n+1)$-Engel element of G.

To continue with the induction, we assume for $2 \le k \le n$ that $[a^{g_1}, \ldots, a^{g_{k-1}}, {}_{n+1-(k-1)} a] = 1$ for all $g_1, \ldots, g_{k-1} \in G$. For simplicity, let $v = [a^{g_1}, \ldots, a^{g_{k-1}}]$. Now $v \in \gamma_{k-1}(G)$, and $t - (n+1-k) = (t-n) + k - 1 < k - 1$, so $v \in \gamma_{(t-n)+k-1}(G)$. Then by Lemma 1.1 and repeated use of Lemma 6.4, again using the fact that $n > t$, there exist $w_1, w_2 \in G'$ such that

$$
\begin{aligned}
[a^{g_1}, \ldots, a^{g_k}, {}_{n+1-k} a] &= [v, a^{g_k}, {}_{n+1-k} a] \\
&= [v, a[a, g_k], {}_{n+1-k} a] \\
&= [[v, [a, g_k]][v, a][v, a, [a, g_k]], {}_{n+1-k} a] \\
&= [v, [a, g_k], {}_{n+1-k} a]^{w_1} \cdot [[v, a][v, a, [a, g_k]], {}_{n+1-k} a] \\
&= [v, [a, g_k], {}_{n+1-k} a]^{w_1} \cdot [v, a, {}_{n+1-k} a]^{w_2} \\
&\quad \cdot [v, a, [a, g_k], {}_{n+1-k} a] \\
&= [v, [a, g_k], {}_{n+1-k} a]^{w_1} \cdot [v, {}_{n+2-k} a]^{w_2} \\
&\quad \cdot [v, a, [a, g_k], {}_{n+1-k} a].
\end{aligned} \tag{133}
$$

Now we examine each commutator in the expansion of $[v, a^{g_k}, {}_{n+1-k} a]$. Since $[v, {}_{n+2-k} a] = [v, {}_{n+1-(k-1)} a]$, the induction hypothesis shows that $[v, {}_{n+2-k} a]$ is trivial. Since $k \ge 2$, we conclude that $[v, a, [a, g_k], {}_{n+1-k} a]$ is trivial by the hypothesis that G is a $\mathfrak{W}_{t+3}$ group. Hence $[a^{g_1}, \ldots, a^{g_k}, {}_{n+1-k} a] = [v, [a, g_k], {}_{n+1-k} a]^{w_1}$. If $3 \le k \le n$, then $[v, [a, g_k], {}_{n+1-k} a]$ is trivial because G is a $\mathfrak{W}_{t+3}$ group. If $k = 2$, then $v = a^{g_1}$, so by Lemma 6.4 and Lemma 6.5, there exist $w, w_1, w_2 \in G'$ such that

$$
\begin{aligned}
[a^{g_1}, [a, g_2], {}_{n-1}a] &= [a[a, g_1], [a, g_2], {}_{n-1}a] \\
&= [a, [a, g_2], {}_{n-1}a]^w \cdot [[a, g_1], [a, g_2], {}_{n-1}a] \\
&= [[[g_2, a]^{-1}, a]^{-1}, {}_{n-1}a]^w \cdot [a, g_1, [a, g_2], {}_{n-1}a] \\
&= [[g_2, a]^{-1}, a, {}_{n-1}a]^{-w_1 w} \cdot [a, g_1, [a, g_2], {}_{n-1}a] \\
&= [g_2, a, a, {}_{n-1}a]^{w_2 w_1 w} \cdot [a, g_1, [a, g_2], {}_{n-1}a] \\
&= [g_2, {}_{n+1}a]^{w_2 w_1 w} \cdot [a, g_1, [a, g_2], {}_{n-1}a].
\end{aligned}
\tag{134}
$$

However, $[g_2, {}_{n+1}a] = 1$ because a is a left $(n+1)$-Engel element of G, and $[a, g_1, [a, g_2], {}_{n-1}a]$ is trivial because G is a $\mathfrak{W}_{t+3}$ group.

We have shown that $[a^{g_1}, \dots, a^{g_k}, {}_{n+1-k}a] = 1$, which completes the induction. Thus $[a^{g_1}, \dots, a^{g_n}, a] = 1$ for $k = n$ which, by Lemma 6.3, completes the proof.

□

The following theorem, which is the major result of this chapter, is a generalization of Theorem 1.4 ([**7**]).

THEOREM 6.7. *Let m be an integer greater than or equal to 4. If G is a $\mathfrak{W}_m$ group, then for $n \geq m-2$, the following are equivalent:*

(1) *x^G is nilpotent of class at most n for every $x \in G$.*

(2) *x^G is n-Engel for every $x \in G$.*

(3) *G is $(n+1)$-Engel.*

PROOF. We know (1) implies (2) and (2) implies (3) by Lemma 1.3. If G is an $(n+1)$-Engel group, then every $x \in G$ is a left $(n+1)$-Engel element, so Lemma 6.6 then shows that x^G is nilpotent of class at most n for every $x \in G$. □

In Chapter 3, we showed that Condition $\mathfrak{E}_n$ and Condition $\mathfrak{L}(\mathfrak{N}_{n-1})$ are not equivalent, and in Chapter 5, we showed that Condition $\mathfrak{E}_n$ and Condition $\mathfrak{L}(\mathfrak{E}_{n-1})$ are not equivalent. However, in this chapter we showed that these conditions are equivalent in the case of $\mathfrak{W}_m$ groups if m is at most $n+2$. As we ask in Questions 6 and 7, it is unknown if Condition $\mathfrak{B}_{n-1}$ is also equivalent to Conditions $\mathfrak{E}_n$, $\mathfrak{L}(\mathfrak{N}_{n-1})$, and $\mathfrak{L}(\mathfrak{E}_{n-1})$ for $\mathfrak{W}_m$ groups if m is at most $n+2$.

Bibliography

[1] W.P. Kappe. Die a-Norm Einer Gruppe. *Illinois J. Math.*, 5:187–197, 1961.

[2] L.-C. Kappe and W.P. Kappe. On three-Engel groups. *Bull. Austral. Math. Soc.*, 7:391–405, 1972.

[3] Narain Gupta and Frank Levin. On soluble Engel groups and Lie algebras. *Archiv der Mathematik*, 34:289–295, 1980.

[4] Michael Vaughan-Lee. On 4-Engel groups. *LMS J. Comput. Math.*, 10:341–353, 2007.

[5] Werner Nickel. Computation of nilpotent Engel groups. *J. Austral. Math. Soc. (ser. A)*, 67:214–222, 1999.

[6] Eliyahu Rips and Aner Shalev. The Baer condition for group algebras. *J. Algebra*, 140(1):83–100, June 1991.

[7] Robert Fitzgerald Morse. Solvable Engel groups with nilpotent normal closures. In C. M. Campbell, E. F. Robertson, N. Ruskuc, and G. C. Smith, editors, *Groups St. Andrews 1997 in Bath, II*, volume 261 of *London Mathematical Society Lecture Note Series*, pages 560–567, Bath, UK, 1999. London Mathematical Society, University of Bath, others, Cambridge University Press.

[8] W. Burnside. On groups in which every two conjugate operations are permutable. *Proc. London Math. Soc.*, 35:28–35, 1902.

[9] William Benjamin Fite. On metabelian groups. *Trans. Amer. Math. Soc.*, 3(3):331–353, July 1902.

[10] W.B. Fite. Groups of order 3^m in which every two conjugate operations are permutable. *Mathematische Annalen*, 67:498–510, 1909.

[11] Nathan Jacobson. Rational methods in the theory of Lie algebras. *The Annals of Mathematics (2)*, 36(4):857–881, Oct. 1935.

[12] C. Hopkins. Finite groups in which conjugate operations are commutative. *American Journal of Mathematics*, 51(1):35–41, Jan. 1929.

[13] L. Paul Bouton. On n-Engel elements: The Macdonald counterexample and beyond. Master's thesis, State University of New York at Binghamton, 1994.

[14] Wilhelm Magnus. Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring. *Math. Ann.*, 111:259–280, 1935.

[15] Wilhelm Magnus. Über Beziehungen zwischen höheren Kommutatoren. *J. Reine und Agnew. Math.*, 177:105–115, 1937.

[16] Wilhelm Magnus. Über Gruppen und zugeordnete Liesche Ringe. *J. Reine und Agnew. Math.*, 182:142–149, 1940.

[17] H. Zassenhaus. Über Lie'sche Ringe mit Primzahl charakteristik. *Abh. Math. Sem. Univ. Hamburg*, 13:1–110, 1940.

[18] Abstracts of papers. *Bull. Amer. Math. Soc.*, 42:483–499, 1936.

[19] Max Zorn. Nilpotence of finite groups. Conference talk, abstract in [**18**], 1936.

[20] Max Zorn. On a theorem of Lie. Conference talk, abstract in [**18**], 1936.

[21] Max Zorn. On a theorem of Engel. *Bull. Amer. Math. Soc.*, 43:401–404, 1937.

[22] Reinhold Baer. Nilpotent groups and their generalizations. *Trans. Amer. Math. Soc.*, 47(3):393–434, May 1940.

[23] N. Gupta. Burnside groups and related topics. Privately distributed lecture notes, Univ. of Manitoba, August 1976.

[24] P. M. Cohn. A non-nilpotent Lie ring satisfying the Engel condition and a non-nilpotent Engel group. *Proc. Cambridge Philos. Soc.*, 51(3):401–405, July 1955.

[25] F. W. Levi. Groups in which the commutator operation satisfies certain algebraic conditions. *J. Indian Math. Soc.(N.S.)*, 6:87–97, 1942.

[26] Heinrich Meier-Wunderli. Über die Gruppen mit der identischen Relation $(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, x_1)$ $(n > 3)$. *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 94:211–218, 1949.

[27] K. W. Gruenberg. Two theorems on Engel groups. *Proc. Cambridge Philos. Soc.*, 49:377–380, 1953.

[28] Reinhold Baer. Engelshe Elemente Noetherscher Gruppen. *Math. Annalen*, 133:256–270, 1957.

[29] E. S. Golod. On nil-algebras and residually finite p-groups. *Izv. Acad. Nauk. SSSR Ser. Mat.*, 28:273–276, 1964.

[30] E. S. Golod. On nil-algebras and finitely approximable p-groups. *Amer. Math. Soc. Translations(2)*, 48:103–106, 1965.

[31] M. S. Garaščuk and D. A. Suprunenko. Linear nilgroups. *Dokl. Akad. Nauk BSSR*, 4:407–408, 1960.

[32] R. B. Burns and Yuri Medvedev. A note on Engel groups and local nilpotence. *J. Austral. Math. Soc.*, 64:92–100, 1998.

[33] Yuri Medvedev. On compact Engel groups. *Israel J. Math.*, 135:147–156, 2003.

[34] J. S. Wilson. Two-generator conditions for residually finite groups. *Bull. London Math. Soc.*, 23:239–248, 1991.

[35] Howard Smith. Bounded Engel groups with all subgroups subnormal. *Comm. Alg.*, 30(2):907–909, 2002.

[36] Alireza Abdollahi and Gunnar Traustason. On three-Engel groups. *Proc. Amer. Math. Soc.*, 180(10):2827–2836, 2002.

[37] Peter Crosby and Gunnar Traustason. A remark on the structure of n-Engel groups. *J. Austral. Math. Soc.* to appear.

[38] K. W. Gruenberg. The upper central series in soluble groups. *Illinois J. Math.*, 5:436–466, 1961.

[39] Hermann Heineken. Engelshe Elemente der Länge Drei. *Illinois J. Math.*, 5:681–707, 1961.

[40] Narain Gupta. Third Engel 2-groups are soluble. *Canad. Math. Bul.*, 15:523–523, 1972.

[41] S. Bachmuth and H.Y. Mochizuki. Third Engel groups and the Macdonald-Neumann conjecture. *Bull. Austral. Math. Soc.*, 5:379–386, 1971.

[42] Hermann Heineken. A class of three-Engel groups. *Journal of Algebra*, 17:341–345, 1971.

[43] N.D. Gupta and M.F. Newman. Third Engel groups. *Bull. Austral. Math. Soc.*, 40:215–230, 1989.

[44] Gunnar Traustason. Two generator 4-Engel groups. *Int. J. Algebra and Comp*, 15(2):1–8, 2005.

[45] George Havas and Michael Vaughan-Lee. 4-Engel groups are locally nilpotent. *Internat. J. Algebra Comput.*, 15(4):649–682, 2005.

[46] Gunnar Traustason. A note on the local nilpotence of 4-Engel groups. *Int. J. Algebra and Comp.*, 15(4):757–764, 2005.

[47] Gunnar Traustason. Locally nilpotent 4-Engel groups are Fitting groups. *J. Algebra*, 270:7–27, 2003.

[48] Gunnar Traustason. On 4-Engel groups. *Journal of Algebra*, 178:414–429, 1995.

[49] A. J. Bayes, J. Kautsky, and J. W. Wamsley. Computation in nilpotent groups (application). In *Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973)*, pages 82–89. Lecture Notes in Math., Vol. 372, Berlin, 1974. Springer.

[50] Ju. P. Razmyslov. The Hall-Higman problem. *Izv. Akad. Nauk. SSSR Ser. Mat.*, 42:833–847, 1978.

[51] Ju. P. Razmyslov. On a problem of Hall and Higman. *Math. USSR Izvestija*, 13(1):133–146, 1979.

[52] Patrizia Longobardi and Mercede Maj. Semigroup identities and Engel groups. In C. M. Campbell, E. F. Robertson, N. Ruskuc, and G. C. Smith, editors, *Groups St. Andrews 1997 in Bath, II*, volume 261 of *London Mathematical Society Lecture Note Series*, pages 527–531, Bath, UK, 1999. London Mathematical Society, University of Bath, others, Cambridge University Press.

[53] M. F. Newman, Werner Nickel, and Alice C. Niemeyer. Descriptions of groups of prime-power order. *J. Symbolic computation*, 25:665–682, 1998.

[54] M.F. Newman and E.A. O'Brien. Applications of computers to questions like those of Burnside, ii. *Internat. J. Algebra Comput.*, 6(5):593–605, 1996.

[55] Charles C. Sims. *Computation with Finitely Presented Groups*. Cambridge University Press, 1994.

[56] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.9*, 2006.

[57] Greg Gamble, Werner Nickel, and Eamonn O'Brien. *ANUPQ: ANU p-Quotient - a GAP package, version 3.0*, 2006.

[58] Waterloo Software. *Maple 9.5*, 2004.

[59] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial Group Theory*. Dover Publications Inc., 2 edition, 1976.

[60] George Bergman. The diamond lemma for ring theory. *Advances in Mathematics*, 29:178–218, 1978.

[61] Ken A. Brown and Ken R. Goodearl. *Lectures on Algebraic Quantum Groups*. Birkhäuser Verlag, 2002.

[62] Gunnar Traustason. Engel Lie-algebras. *Quart. J. Math. Oxford(2)*, 44:355–384, 1993.

[63] Marshall Hall. *The Theory of Groups*. Chelsea Publishing Co, 1976.

[64] Derek J. S. Robinson. *A Course in the Theory of Groups*, volume 80 of *Graduate texts in mathematics*. Springer-Verlag, 2 edition, 1996.

[65] I. B. S. Passi, D. S. Passman, and S. K. Seghal. Lie solvable group rings. *Canad. J. Math.*, 25:748–757, 1973.

[66] A. A. Bovdi and I. I. Kripta. Group algebras of torsion groups with a soluble multiplicative group. *Mat. Zametki*, 22:421–432, 1977.

[67] A. A. Bovdi and I. I. Kripta. Group algebras of periodic groups of a solvable multiplicative group. *Math. Notes*, 22(3):725–731, 1977.

[68] César Milies and Sudarshan Sehgal. *An Introduction to Group Rings*. Kluwer Academic Publishers, 2002.

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Christine Bussman was born in 1978 in Glen Ellyn, Illinois. She graduated from Glenbard West High School and received the degree of B.S. from Wheaton College in 1999. She presented at the Missouri meeting of the MAA in 2002 and 2003. She also attended the OSU/Denison/Zassenhaus group theory conference in 2002, 2003, 2005, and 2007. In 2007 she gave a presentation. She also attended a special session on infinite groups and group rings at the 2003 AMS meeting and attended Groups Saint Andrews 2005.