A Sketch of the History of Engel Groups

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1 Early History of Engel Groups

In this section we examine the history of Engel groups in general. In the next section, we consider the history of n-Engel groups in particular.

The groups now called Engel groups have become an area of interest in their own right, but their history is not well known beyond the fact that they are "related" to Lie algebras.

The earliest known reference to groups that are later called Engel groups occurs in the work of Burnside in 1902 ([9]). Burnside's paper refers to "groups where any two conjugate operations commute," Lemma 1 shows he is discussing 2-Engel groups.

Lemma 1. *A group G is a 2-Engel group if and only if any two conjugate elements of G commute.*

Proof. Any two conjugate elements commute in group G exactly when for any $a, b \in G$, we know $[a^b, a] = 1$. However,

$$\begin{aligned} [a^b, a] &= [a[a,b], a] \\ &= [a, b, a] \\ &= [[b,a]^{-1}, a] \\ &= [b, a, a]^{-[b,a]^{-1}}. \end{aligned} \tag{1}$$

Thus $[a^b, a]$ is trivial exactly when $[b, a, a]$ is trivial. □

In 1909, Fite ([12]) showed that 2-Engel groups (using terminology similar to that of Burnside) are nilpotent of class at most 3. By providing an example of a 2-Engel group of class exactly 3, Fite showed that 3 is the best possible bound on the nilpotence class of a 2-Engel group.

It seems that current interest in Engel groups comes from questions about Lie algebras. The first result of note in this direction is an unpublished result of van Kampen that finite dimensional Engel Lie algebras over fields of characteristic 0 are nilpotent. This result is referenced in Jacobson ([27]), who says that it was "found by Dr. van Kampen, Hamburg, 1928, but has not been previously published."

In 1929 Hopkins ([26]) also showed that finite 2-Engel groups are nilpotent of class at most 3. He refers to the work of Burnside ([9]), but not the work of Fite ([12]). His terminology is also of interest because he refers to 2-Engel groups as groups where "conjugate *elements* commute," which is closer to our current usage than that of Burnside.

The same result as that of van Kampen was also proved by Weyl in class lectures (also cited by [27]). Jacobson ([27]) proved that finitely generated Engel Lie algebras over fields of characteristic 0 are nilpotent.

Bouton ([7]) states that Magnus and Zassenhaus drew the connection between Lie algebras and groups in the 1930s. Papers Magnus published during this time frame that include this connection are ([32]), ([33]), and ([34]). Zassenhaus also used this connection in his 1940 paper ([48]), in which he shows that finite Engel groups are nilpotent and in which he connects Lie algebras with finite groups. On page 4 of ([48]), Zassenhaus takes credit for the label "nilpotent" for groups. Zassenhaus also lists and proves some analogous conditions for groups and Lie algebras.

In the abstracts ([1]) for an AMS conference on June 18, 1936, M. Zorn describes two talks that are of interest. In one of these ([49]), he proves that any finite Engel group is nilpotent, and in the other ([50]), that a finite Engel Lie algebra over a field of characteristic 0 is nilpotent (he refers erroneously to this result as related to a theorem of Lie, rather than of Engel, as he points out later ([51])). In Zorn's work the connections between Lie algebras and groups are also apparent, because the concepts of "Engel" and "nilpotent" are used for groups, although they were first used for rings and algebras. Zorn explicitly says in the abstract about groups ([49]) that the group abstract is the analogue of the abstract about algebras ([50]). The next year, he published another paper ([51]) in which he shows that n-Engel Lie rings with the maximal condition on subrings are nilpotent. He comments that this paper replaces one of the abstracts from the previous year.

In 1940, Baer ([4]) provided examples of non-nilpotent n-Engel groups. Baer does not explicitly state that his examples have these properties, but he does state that he provides for each prime p an infinite p-group that has trivial center but abelian commutator subgroup. These groups (Example 3.4 in [4]) are obtained by manipulating p-adic expansions of integers, but as Gupta points out ([19], page 32) they are equivalent to wreath products $C_p \wr (\oplus_{i=1}^{\infty} C_p)$. Cohn ([10]) notes that Baer's examples are n-Engel but not nilpotent.

In 1942, Levi ([30]) proved the following:

Theorem 2. *Let G be a 2-Engel group. Then*

1. *G is nilpotent of class ≤ 3.*
2. *If a, b, and c are elements of G, then $[a, b, c]^3 = 1$.*
3. *$[a, b, c] = [b, c, a]$ for all a, b, c elements of G.*

The first part of Theorem 2 was already shown for finite groups in ([26]), but it is Levi's result that is usually cited now. Meier-Wunderli ([36]) proved these same results in 1949 and pointed to Fite's example of a 2-Engel group of class exactly 3 to show that 3 is the best possible bound on the nilpotence class of a 2-Engel group.

In 1953 Gruenberg ([17]) showed that a finitely generated solvable Engel group is nilpotent, and that a finitely generated solvable Engel Lie ring is nilpotent. Gruenberg's paper is of particular note because it is the earliest paper the author of this sketch can locate in which the term "Engel group" is used. M. Zorn and others before this discuss Engel groups, but do not use the term "Engel group." Cohn ([10]) gives Gruenberg credit for first using the term "Engel group," but does not specify where he does so.

In 1954 Cohn ([10]) provided examples of a non-nilpotent Engel group and a non-nilpotent Engel ring. He credits Gruenberg with an unpublished example of such a group (his exact wording is "Dr. Gruenberg has informed me that he also has such an example"), and adds a note that after writing his paper he was made aware of Baer's 1940 example. Cohn starts with F a field of characteristic p, where p is a prime, and then constructs K, a purely transcendental extension of F of countable (infinite) transcendence degree. He then constructs M, a free left K-module on an infinite basis, and considers K to be the free K-module on the generator 1, where 1 is the unit element of K. The associative algebra A is then constructed as a sum $K + M$, and Cohn imposes a Lie multiplication on A, getting a Lie algebra L. He shows that L is $(p+1)$-Engel, but not nilpotent. Cohn's group example G is generated by elements of the form $\alpha(1+x)$, where α is a nonzero element of K and x is an element of M. This group is isomorphic to $C_p \wr (\oplus_{i=1}^{\infty} Cp)$. The group G is $(p+1)$-Engel but not nilpotent.

In 1957, Baer ([5]) defined a group to be Noetherian if every subgroup is finitely generated, and proved that Engel groups that are Noetherian are also nilpotent.

Golod ([15]) (translated [16]) constructed a p-group that is finitely generated, residually finite, and not locally nilpotent. He does so by considering the multiplicative group generated by $\{1+x_i\}$, where the x_i are generators of a particular polynomial ring over the field $\mathbb{Z}_p$. He comments that the same method of construction can be used to give a non-nilpotent finitely generated Engel group, thus proving that not all Engel groups are locally nilpotent.

To add to the information about Engel $\mathfrak{X}$-groups for various group-theoretic properties $\mathfrak{X}$, in 1960 Garaščuk and Supranenko ([14]) showed that a linear group over a field of characteristic zero is Engel if and only if it is locally nilpotent.

In 1998, Burns and Medvedev ([8]) showed that the commutator subgroup of a finitely generated Engel group is also finitely generated.

In 2003 Medvedev ([35]) proved that a compact Engel group is locally nilpotent.

2 Recent History of n-Engel Groups

In this section we examine more recent historical developments in the study of n-Engel groups, organized by property. The organization of this section is topical rather than chronological.

2.1 General n-Engel Groups

In 1991 Wilson ([47]) showed that a finitely generated residually finite n-Engel group is nilpotent.

In 2002, H. Smith ([41]) proved that every n-Engel group in which every subgroup is subnormal is nilpotent. In the same year, Abdollahi and Traustason ([2]) showed that for every positive integer n, there is a positive integer $s(n)$ such that every powerful n-Engel p-group is nilpotent of class at most $s(n)$.

Traustason and Crosby will soon publish a paper ([11]) in which they show that there exist positive integers $m(n)$ and $r(n)$ such that the law $[x^{r(n)}, x_1, \ldots, x_{m(n)}] = 1$ holds for all x and for all $x_i, i = 1, \ldots, m(n)$ in any locally nilpotent n-Engel group.

In 1961, Gruenberg ([18]) showed that every n-Engel group that is solvable of derived length d and that has no elements of prime order less than n is nilpotent of class at most $(1+2^{n-2})^d/2^{n-2}$.

2.2 One-Engel Groups

A group that is 1-Engel is abelian. Abelian groups have their own theory that is not in the scope of this sketch.

2.3 Two-Engel Groups

A 2-Engel group is, in some sense, almost abelian. In 1961, W.P. Kappe ([29]) showed that if a is an element of a 2-Engel group G, then a^G, the normal closure of a in G, is abelian. We pointed out earlier (Theorem 2) that any 2-Engel group is nilpotent of class at most 3, and is nilpotent of class at most 2 if there are no elements of exponent 3 in the group by part (2) of Theorem 2. Other important papers that discuss 2-Engel groups are by Burnside ([9]), Fite ([13]) ([12]), and Levi ([30]), and are discussed in Section 1 above.

2.4 Three-Engel Groups

Heineken ([24]) proved that all 3-Engel groups are locally nilpotent, and that if a 3-Engel group G has no elements of order 2 or 5, then G is nilpotent of class at most 4. If G has an element of order 2 but not of order 5, then G is solvable (N. Gupta, [20]) but not necessarily nilpotent (Baer, [4]). Bachmuth and Mochizuki ([3]) in 1971 gave an example of a 3-Engel group of exponent 5 that is not solvable, let alone nilpotent.

Heineken ([25]) in 1971 described a subclass of 3-Engel groups that have properties like those described by Levi and others for 2-Engel groups. Heineken's paper considers those 3-Engel groups whose cyclic subgroups are subnormal of defect 2, and shows that these groups are nilpotent of class at most 5, that weight 4 commutators in these groups have exponent 3, and that $[a, [b, [b, c]]] = 1$ for all elements a, b, and c of such groups. All 2-Engel groups satisfy all three of these conditions.

In ([28]) W.P. Kappe and L.-C. Kappe showed that G is a 3-Engel group if and only if for every element a of G, the nilpotence class of a^G is 2 or smaller, which he also showed happens if and only if for every element a of G, the normal closure a^G is 2-Engel. In 1989 N. Gupta and Newman ([22]) showed that a 3-Engel n-generator group G is nilpotent of class at most $2n - 1$ if $n > 2$, that a 3-Engel group with no elements of order 5 has nilpotence class at most $n+2$, and that the exponent of the 5th term of the lower central series of a 3-Engel group divides 20. Heineken in 1961 ([24]) showed that a 3-Engel group that has no elements of order 2 or 5 is nilpotent of class at most 4.

2.5 Four-Engel Groups

There are many results about 4-Engel groups, and for this reason the results in this subsection are given ordered by importance for the author's interests.

In 2005, Traustason ([45]) gave some properties of 2-generator 4-Engel groups. Later in 2005, Havas and Vaughan-Lee ([23]) proved that all 4-Engel groups are locally nilpotent, although their proof was highly dependent on computer computations in a specific group. In that same year, Traustason ([44]) published results that replace Havas and Vaughan-Lee's computer calculations with hand calculations, effectively making the proof of Havas and Vaughan-Lee computer free.

In 1980, Gupta and Levin ([21]) constructed a 4-Engel 5-group that has an element whose normal closure is not nilpotent of class 3. In 1999, Nickel ([37]) constructed a 2-group example similar to the example of Gupta and Levin for 5-groups. In 2003, Traustason ([43]) proved that the normal closure of every element of a 4-Engel group without elements of order 2 or 5 is nilpotent of class 3. In 2007, Vaughan-Lee ([46]) proved that the normal closure of every element of a 4-Engel group is 3-Engel.

Traustason ([42]) showed in 1994 that if a 4-Engel group has no elements of order 2, 3, or 5, then it is nilpotent of class at most 7. In 2002, Abdollahi and Traustason ([2]) showed that a 4-Engel group without elements of order 2 or 5 is solvable, but a 4-Engel group with elements of order 5 is not always solvable because Bachmuth and Mochizuki ([3]) construct a 3-Engel group with elements of order 5 that is not solvable. Abdollahi and Traustason ([2]) point out that a 4-Engel 2-group is not necessarily solvable because every group of exponent 4 is central-by-4-Engel (Bayes, Kautsky, and Wamsley [6]) and Razmyslov ([38]) (translated into English as [39]) constructs a non-solvable group of exponent 4 (note that if a group is not solvable, then the quotient of that group by its center remains non-solvable).

Longobardi and Maj ([31]) proved in 1997 that every right orderable 4-Engel group is nilpotent.

2.6 n-Engel Groups for Larger Values of n

In ([46]), Vaughan-Lee constructed a 5-Engel group that has an element whose normal closure is not 4-Engel. Vaughan-Lee's example is a 3-group. It is not known if for other primes p there exist a 5-Engel p-groups that contains an element whose normal closure is not 4-Engel. It is also not known if 5-Engel groups are locally nilpotent. In the author's dissertation, it is shown that there are 2-groups, 3-groups, 5-groups, and 7-groups that are 5-Engel but not every normal closure is nilpotent of class 4.

For values of n greater than 5, groups that are n-Engel have not been extensively studied. The author's dissertation showed that there are groups that are 6-Engel but contain an element whose normal closure is not nilpotent of class 5. Rips and Shalev ([40]) show that for every odd prime p, there is some value of n such that for p-groups, there is an n-Engel p-group in which an element has a normal closure that is not $(n-1)$-Engel. Gupta and Levin ([21]) also show that for every odd prime p, there is a p-group that is $(p+2)$-Engel but which has an element whose normal closure is not nilpotent. It is not known if n-Engel groups are locally nilpotent for values of n larger than 5.

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